Huygens-Kirchhoff's Theory in Calculation of Elliptical Gaussian Beam Propagation through a Lens

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Abstract. **Based on Huygens-Kirchhoff's theory and thin lens phase change formulation, new closed form expressions have been derived for the radiation field of an elliptical Gaussian beam transmitted through a thin lens. For a special case of paraxial approximation, the new expressions reduce to well known q-parameter results if applied separately to the *x*- and *y*-Gaussian field expressions. As an example of using presented theory the condition to convert elliptical to quazi-circular Gaussian beam is derived.**
Key words: **Elliptical Gaussian beam, transmission, lens**

I. INTRODUCTION

Circular Gaussian field propagation through space and its transformation through lenses can be made using q-parameter and its transformation matrices. The radiation field can also be obtained as the solution of the paraxial wave equation [1], or as the solution of radiation field from an illuminated aperture [2]. All these are normally applied to a circular Gaussian beam and are given in closed form solutions. In the case of an elliptical Gaussian beam, closed form solution can also be obtained under assumption that separable solutions to the *x*- and *y*- field functions is applicable [2].

In this work we calculate Gaussian beam radiated from an aperture illuminated by an elliptical field distribution using Huygens-Kirchhoff's theory. By applying paraxial field approximation, the radiation field double integral reduces to the product of two line integrals with respect to the *x*- and *y*-components.

II. BASIC THEORY

The aperture field of an elliptical Gaussian beam is assumed in the form:

$$E(x',y',0)=E_0\exp\left(-\frac{x'^2}{w_{0x}^2}-\frac{y'^2}{w_{0y}^2}\right) \tag{1}$$

where w_{0x} and w_{0y} are the beam half-widths or beam waists along the *x*- and *y*- axes, respectively. Starting from (1) as the aperture field, and applying Huygens-Kirchhoff's method, under the condition that the distance between the aperture elementary area *dx'dy', at* the point *(x',y')*, and the field point

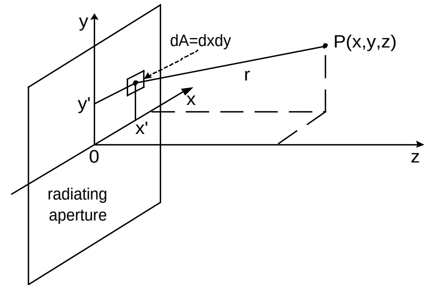

Fig.1. Geometry used in calculation of Gaussian beam radiation

(x,y,z), as shown in Fig.1, can be simplified under the paraxial (or far field) assumption

$$r=z+\frac{(x-x')^2}{2z}+\frac{(y-y')^2}{2z} \tag{2}$$

Under this condition the field in front of the aperture can be obtained in a closed form [2]:

$$E(x,y,z)=E_z\exp[-j(kz-\psi)]\times$$
$$\times\exp\left[-\left(\frac{1}{w_{zx}^2}+j\frac{\pi}{\lambda R_{zx}}\right)x'^2-\left(\frac{1}{w_{zy}^2}+j\frac{\pi}{\lambda R_{zy}}\right)y'^2\right] \tag{3}$$

where $E_z=\sqrt{\dfrac{w_{0x}w_{0y}}{w_{zx}w_{zy}}}E_0$,

$\psi=\frac{1}{2}\tan^{-1}\left(\lambda z/\pi w_{0x}^2\right)+\frac{1}{2}\tan^{-1}\left(\lambda z/\pi w_{0y}^2\right)$, and

$$w_{zx}=w_{0x}\sqrt{1+(\lambda z/\pi w_{0x}^2)^2},$$
$$w_{zy}=w_{0y}\sqrt{1+(\lambda z/\pi w_{0y}^2)^2}$$
$$R_{zx}=\left(1+(\pi w_{0x}^2/\lambda z)^2\right),\ R_{zy}=\left(1+(\pi w_{0y}^2/\lambda z)^2\right)$$

are the 1/e half-widths and the radius of curvatures along the *x*- and *y*- axes, respectively.

Thin convex lens is assumed to change only the phase of the wave given by Eq. (3) and to get the field after the lens, one has to multiply it by the function [1]:

$$\exp\left[j\frac{k}{2f}(x^2+y^2)\right] \tag{4}$$

where *f* is the focal length of the convex lens and $k=2\pi/\lambda$.

In calculating the radiation field from the lens we assume that the aperture field is given by Eq.(3) for $z=z_1$, multiplied by

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the expression (4). The aperture coordinates are now (ξ,η) and the far field point is again *(x,y,z)*, but the coordinate *z* is now measured from the aperture centre. Now we apply the Huygens-Kirchhoff's integral to find the radiation field

$$E_{rad}(x,y,z)=\frac{j}{\lambda}\int_A E_{ap}(\xi,\eta,0)\frac{e^{-jkr}}{r}dA \tag{5}$$

where *r* in the exponential is taken as given by Eq.(2) and as *r*=*z* in the denominator. The double integral in Eq.(5) can be split into the product of two line integrals of the form (shown only for the *x*-component):

$$I_x=\int_{-\infty}^{\infty}\exp\left\{-\left[\frac{1}{w_{z1x}^2}+j\frac{\pi}{\lambda}\left(\frac{1}{R_{z1x}}-\frac{1}{f}\right)\right]\xi^2-j\frac{\pi}{\lambda z}(x-\xi)^2\right\}d\xi \tag{6}$$

The above integral can be further simplified

$$I_x(x,z)=\exp\left[-\left(\frac{\pi^2}{\lambda^2 z^2 a_x^2}+j\frac{\pi}{\lambda z}\right)x^2\right]\times \times\int_{-\infty}^{\infty}\exp\left[-\left(a_x\xi-j\frac{\pi x}{\lambda z a_x}\right)^2\right]d\xi \tag{7}$$

and after the change of variable

$$a_x\xi-j\frac{\pi x}{\lambda z a_x}=u$$

it is reduced to a table integral and solved to give finally:

$$I_x(x,z)=\frac{\sqrt{\pi}}{a_x}\exp\left[-\left(\frac{\pi^2}{\lambda^2 z^2 a_x^2}+j\frac{\pi}{\lambda z}\right)x^2\right] \tag{8}$$

where

$$a_x^2=\frac{1}{w_{z1x}^2}+j\frac{\pi}{\lambda}\left(\frac{1}{R_{z1x}}-\frac{1}{f}+\frac{1}{z}\right) \tag{9}$$

In a similar way the integral with respect to *y* leads to

$$I_y(y,z)=\frac{\sqrt{\pi}}{a_y}\exp\left[-\left(\frac{\pi^2}{\lambda^2 z^2 a_y^2}+j\frac{\pi}{\lambda z}\right)y^2\right] \tag{10}$$

where

$$a_y^2=\frac{1}{w_{z1y}^2}+j\frac{k}{2}\left(\frac{1}{R_{z1y}}+\frac{1}{z}-\frac{1}{f}\right) \tag{11}$$

Finally, the radiation field behind the lens is from Eq.(3,5,8,10)

$$E_{rad}(x,y,z)=\frac{jE_{z1}}{\lambda z a_x a_y}e^{-j(kz-\psi)} \times\exp\left[-\left(\frac{k^2}{4a_x^2z^2}+j\frac{k}{2z}\right)x^2-\left(\frac{k^2}{4a_y^2z^2}+j\frac{k}{2z}\right)y^2\right] \tag{12}$$

where $k=2\pi/\lambda$.

The radiated field is also Gaussian and the 1/e half width at some distance z can be found after separating the real part of the term

$$\frac{k^2}{4z^2\left[\frac{1}{w_{z1x}^2}+j\frac{k}{2}\left(\frac{1}{R_{z1x}}+\frac{1}{z}-\frac{1}{f}\right)\right]}= =\frac{\frac{k^2}{w_{z1x}^2}-j\frac{k}{2}\left(\frac{1}{R_{z1x}}+\frac{1}{z}-\frac{1}{f}\right)}{4z^2\left[\frac{1}{w_{z1x}^4}+\frac{k^2}{4}\left(\frac{1}{R_{z1x}}+\frac{1}{z}-\frac{1}{f}\right)^2\right]} \tag{13}$$

The required square of the 1/e half-widths are then for the *x*-component

$$w_{zfx}^2=\frac{4z^2w_{z1x}^2}{k^2}\left[\frac{1}{w_{z1x}^4}+\frac{k^2}{4}\left(\frac{1}{R_{z1x}}+\frac{1}{z}-\frac{1}{f}\right)^2\right]= =w_{z1x}^2\left(1+\frac{z}{R_{z1x}}-\frac{z}{f}\right)^2+\frac{z^2\lambda^2}{\pi^2w_{z1x}^2} \tag{14}$$

and for the *y*- component

$$w_{zfy}^2=w_{z1y}^2\left(1+\frac{z}{R_{z1y}}-\frac{z}{f}\right)^2+\left(\frac{z\lambda}{\pi w_{z1y}}\right)^2 \tag{15}$$

For *z*=0 we have $w_{0fx}=w_{z1x}$ and $w_{0fy}=w_{z1y}$ as it should be. The position of the beam waist after the lens is obtained from the derivative of (14) or (15). For the *x*-component it is

$$z_{\min(x)}=\frac{w_{z1x}^2\left(\frac{1}{f}-\frac{1}{R_{z1x}}\right)}{w_{z1x}^2\left(\frac{1}{f}-\frac{1}{R_{z1x}}\right)^2+\left(\frac{\lambda}{\pi w_{z1x}}\right)^2} \tag{16}$$

After substitution of w_{z1x} and R_{z1x} from Eq.(3) into Eq.(16), identical expression as the one obtained by the q-parameter calculation is found:

$$z_{\min(x)}=\frac{z_1(z_1-f)+a_{0x}^2}{(z_1-f)^2+a_{0x}^2}f \tag{16'}$$

For the *y*-component similar calculation leads to

$$z_{\min(y)} = \frac{w_{z1y}^2\left(\frac{1}{f}-\frac{1}{R_{z1y}}\right)}{w_{z1y}^2\left(\frac{1}{f}-\frac{1}{R_{z1y}}\right)^2+\left(\frac{\lambda}{\pi w_{z1y}}\right)^2} = \tag{17}$$

$$= \frac{z_1(z_1-f)+a_{0y}^2}{(z_1-f)^2+a_{0y}^2} f$$

Upon substitution of the beam waist position from Eq.(16) into Eq.(14) it is found that the waist is

$$\frac{1}{w_{wx}^2} = \frac{1}{w_{z1x}^2} + \left(\frac{\pi w_{z1x}}{\lambda}\right)^2\left(\frac{1}{R_{z1x}}-\frac{1}{f}\right)^2 = \tag{18}$$

$$= \frac{(z_1-f)^2+a_{0x}^2}{w_{0x}^2 f^2}$$

which is identical to the one derived on the basis of q-parameter for the *x*- component. The y- component is obtained in a similar way and the only difference is the index in (18) "*x*" which has to be replaced by "*y*".

After further elaborate calculations for the amplitude at the axis one obtains

$$\left|E_{rad}(x,y,z)\right| = \sqrt{\frac{w_{0x}w_{0y}}{w_{zfx}w_{zfy}}} E_0 \tag{19}$$

where w_{zfx} and w_{zfy} are obtained from Eq.(14) and (15), respectively.

III. TRANSFORMATION OF ELLIPTICAL BEAM

In the case when paraxial assumption holds, it is possible to find easily parameters of the elliptical beam after the lens. The most interesting point is that *x*- and *y*- field integrals given by Eqs.(8) and (10), after passing through a lens will have, in general, different change with distance. This results in a non-spherical wave front, and different waists w_{1x} and w_{1y}, as well as different waist positions z_{2x} and z_{2y}, as shown in Fig.2.

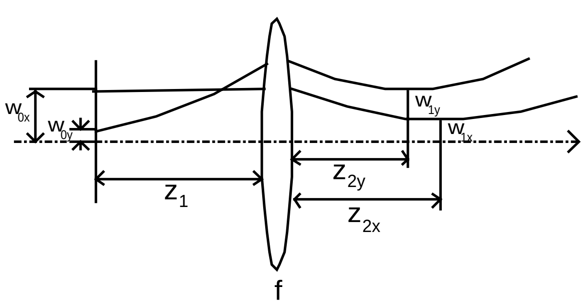

Fig.2. Variation of 1/e half-widths with distance, before and after passing through a convex lens.

For simplicity, in Fig.2, variations of w_{zx} and w_{zy}, are shown in one plane although they are in the orthogonal planes. The beam waists after the lens are

$$w_{1x}^2 = \frac{w_{0x}^2 f^2}{(z_1-f)^2+a_{0x}^2} \tag{20}$$

$$w_{1y}^2 = \frac{w_{0y}^2 f^2}{(z_1-f)^2+a_{0y}^2} \tag{21}$$

In general, w_{1x} and w_{1y} are different, because a_{0x}, w_{0x} and a_{0y}, w_{0y} are different, but we found that by varying z_1, it is possible to find condition that leads to $w_{1x} = w_{1y}$. After simple calculations and using substitutions $a_{0x} = \pi w_{0x}^2/\lambda$, and $a_{0y} = \pi w_{0y}^2/\lambda$, it was found that the condition of equal beam waists is fulfilled for

$$z_1 = f \pm \frac{\pi w_{0x} w_{0y}}{\lambda} \tag{22}$$

In that case

$$w_{1x} = w_{1y} = \frac{\lambda f}{\pi\sqrt{w_{0x}^2+w_{0y}^2}} \tag{23}$$

Equal beam-waists means that the beam angles in the *x*- and *y*- planes are equal, and this converts elliptical beam into a quazi-circular beam at large distances.

Distances z_{2x} and z_{2y} are different as can be seen from the expressions

$$z_{2x} = f + \frac{f^2(z_1-f)}{(z_1-f)^2+a_{0x}^2} \tag{24}$$

$$z_{2y} = f + \frac{f^2(z_1-f)}{(z_1-f)^2+a_{0y}^2} \tag{25}$$

In a special case for $z_1 = f$, distances z_{2x} and z_{2y} are equal and equal to the focal distance, but in that case we have different w_{1x} and w_{1y}, and their ratio is

$$\frac{w_{1x}}{w_{1y}} = \frac{w_{0y}}{w_{0x}} \tag{26}$$

Eq.(26) can be interpretted as the conditionto have twisted elliptical beam for 90^0. Since in this case both beam waists are at the same position, the phase front in the focal plane is plane.

In general case, the ratio of beam waists can be calculated from the expression

$$\frac{w_{1x}}{w_{1y}} = \frac{w_{0x}}{w_{0y}} \sqrt{\frac{(z_1 - f)^2 + a_{0y}^2}{(z_1 - f)^2 + a_{0x}^2}} \qquad (27)$$

The two special cases given by Eq.(23) and (26) can be derived from Eq.(27).

IV CONCLUSIONS

In this paper we presented new expressions for elliptical Gaussian beam transformation through a thin convex lens. It is shown that the paraxial approximation allows the double radiation integral conversion to the product of two line integrals. The variable of one line integral is x-, and in the other integral is y-, and this fact allows use of Gaussian beam parameters originally derived for a circular beam. It is also proved that the field at the beam axis can be calculated from $E_0\sqrt{w_{0x}w_{0y}} = E_z\sqrt{w_{zx}w_{zy}}$ which in the circular beam case becomes $E_0 w_0 = E_z w_z$. Both expressions can also be derived from the constant radiation power through an infinite aperture.

Special case of elliptical beam transformation to quazi circular beam, if applicable in practical cases, can be of value in transforming semiconductor laser beam into the circular one. Further research is under way to clarify possibilities of the proposed transformation technique and check for the errors in applying the paraxial approximation. The new approach can be used in calculation of non-separable solution for the radiation field if numerical integration is performed.

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