Analysis of Cascade-Connected Planar Transmission Lines by *ETS* Method

Miodrag Gmitrovi} and Biljana Stojanovi}

Abstract – **Equivalent Thevenin Source (*ETS*) method is here used for analysis of two-dimensional circuit constructed as cascade-connected uniform transmission lines with different lengths and reduced widths. New relations for solving such circuits are given and presented procedure is verified on two examples.**

Keywords – ***ETS* method, planar transmission line, cascade connection, network junction, reduced number of ports.**

I. INTRODUCTION

Different concepts for modelling and analysis of a large class of two-dimensional circuit structures are given in the papers [1-8]. The papers [1-2] describe two different methods for analysis two-dimensional transmission line equivalent circuit. In both papers the line is characterized in term of its transmission matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ to discuss its properties. In the paper [1] *ETS* method is given and analysis is based on decomposition that two-dimensional circuit into cascade-connected ladder subnetworks with same number of input and output ports. This method is much efficient than the method of direct multiplication of the individual chain matrices [2].

The *ETS* method described in [1] can be applied to cascade connection of uniform lines with different lengths and reduced widths. Two-dimensional circuits are represented as cascade-connected networks with different reduced number of ports. Relations needed for analysis of such networks are given here. Also, the proposed procedure is verified on two examples of impedance transformer with different number of sections in cascade and different reduced number of ports in each section.

II. *ETS* VOLTAGE AND IMPEDANCE CALCULATION – A SIMPLE CASE

Cascade-connected planar transmission lines of different widths and lengths can be analysed as cascade-connected networks with different number of input and output ports. Such multi-port complex network is shown in Fig.1, where the first network is *ETS* described by matrix relation

$$\boldsymbol{U}_S = \boldsymbol{U}_{2T} - \boldsymbol{Z}_{2T}\boldsymbol{I}_S \,. \tag{1}$$

In case of real sources the matrix $\boldsymbol{Z}_{2T}$ is a diagonal matrix which elements are

$$z_{i,i} = R_{i,i} \,, \tag{2}$$

and $i = 1,2,...,L$. The first and the other networks including the k^{th} network as the last one have $2L$ ports. The $k+1^{st}$ network and all networks till the end have $2L_1$ ports, where $L_1 < L$. The voltage and current's subscripts 1 and 2 indicate the input and output ports, respectively, and the superscript indicates the number of network in cascade connection.

Cascade-connected networks with the same number of input and output ports can be analysed by *ETS* method [1]. Voltages and impedances of the k^{th} network can be recovered from the recurrent relations

$$\boldsymbol{U}_{2T}^{k} = [\boldsymbol{A}_k + \boldsymbol{Z}_{2T}^{k-1}\boldsymbol{C}_k]^{-1}\boldsymbol{U}_{2T}^{k-1} \,, \tag{3}$$

$$\boldsymbol{Z}_{2T}^{k} = [\boldsymbol{A}_k + \boldsymbol{Z}_{2T}^{k-1}\boldsymbol{C}_k]^{-1}[\boldsymbol{B}_k + \boldsymbol{Z}_{2T}^{k-1}\boldsymbol{D}_k] \,. \tag{4}$$

It can be shown that these recurrent relations can be used also for the next cascade-connected network with different reduced number of ports.

Multi-port network with L input and L output ports can be described by the equation system

$$\boldsymbol{U}_1^k = \boldsymbol{A}_k\boldsymbol{U}_2^k + \boldsymbol{B}_k\boldsymbol{I}_2^k \,, \tag{5}$$

$$\boldsymbol{I}_1^k = \boldsymbol{C}_k\boldsymbol{U}_2^k + \boldsymbol{D}_k\boldsymbol{I}_2^k \,, \tag{6}$$

where $\boldsymbol{A}_k$, $\boldsymbol{B}_k$, $\boldsymbol{C}_k$ and $\boldsymbol{D}_k$ are transmission matrices of the k^{th} network in the cascade connection. If those networks are ladder networks then their matrices are given in the paper [1].

The impedance matrix obtained by the equation (4) is full matrix

$$\boldsymbol{Z}_{2T}^{k} = \begin{bmatrix} Z_{11}^k & \cdots & Z_{1L}^k \\ \vdots & \ddots & \vdots \\ Z_{L1}^k & \cdots & Z_{LL}^k \end{bmatrix} \tag{7}$$

and it can be divided according the network connection given in Fig.2 in the next form

$$\boldsymbol{Z}_{2T}^{k} = \left[\begin{array}{c|c|c} \boldsymbol{Z}_{11}^k & \boldsymbol{Z}_{12}^k & \boldsymbol{Z}_{13}^k \\ \hline \boldsymbol{Z}_{21}^k & \boldsymbol{Z}_{22}^k & \boldsymbol{Z}_{23}^k \\ \hline \boldsymbol{Z}_{31}^k & \boldsymbol{Z}_{32}^k & \boldsymbol{Z}_{33}^k \end{array}\right]\begin{array}{l} 1 \\ m-1 \\ m \\ n \\ n+1 \\ L \end{array} \,. \tag{8}$$

$$1 \quad m\text{-}1 \,|\, m \qquad n \,|\, n+1 \quad L$$

The output voltage vector of the k^{th} network can be divided in the next form

$$\begin{aligned} \boldsymbol{U}_2^k &= \left[U_{2,1}^k \cdots U_{2,m-1}^k \,|\, U_{2,m}^k \cdots U_{2,n}^k \,|\, U_{2,n+1}^k \cdots U_{2,L}^k\right]^T \\ &= \left[\boldsymbol{U}_{21}^k \,|\, \boldsymbol{U}_{22}^k \,|\, \boldsymbol{U}_{23}^k\right]^T \end{aligned} \tag{9}$$

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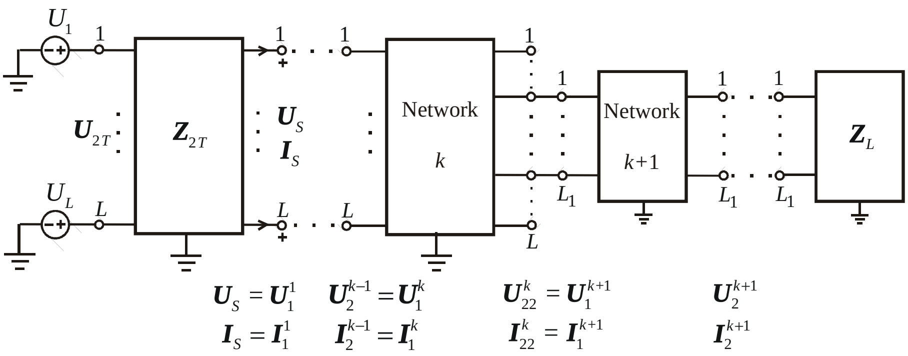

Fig.1. Cascade connection of networks with different number of ports.

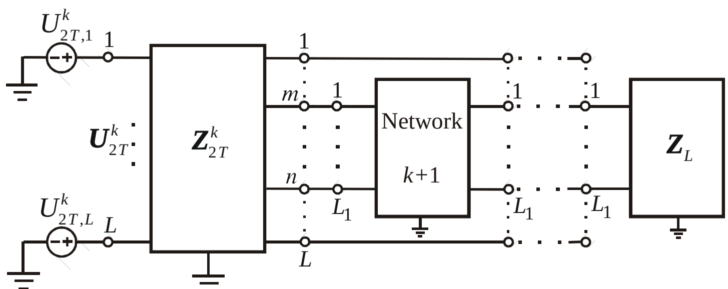

Fig.2. Junction of two networks with different number of ports.

and the voltage vector of *ETS* at the k^{th} open ended network can be divided in the form

$$\boldsymbol{U}_{2T}^k = \left[U_{2T,1}^k \cdots U_{2T,m-1}^k \mid U_{2T,m}^k \cdots U_{2T,n}^k \mid U_{2T,n+1}^k \cdots U_{2T,L}^k\right]^T = \left[\boldsymbol{U}_{2T,1}^k \mid \boldsymbol{U}_{2T,2}^k \mid \boldsymbol{U}_{2T,3}^k\right]^T \tag{10}$$

The transmission matrices of the $k+1^{st}$ network and the other networks till the end are quadratic matrices of sizes $L \times L$ and shapes

$$\boldsymbol{A}_{k+1} = \begin{bmatrix} \boldsymbol{1} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_{k+1}^r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{1} \end{bmatrix}, \quad \boldsymbol{B}_{k+1} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B}_{k+1}^r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \tag{11}$$

$$\boldsymbol{C}_{k+1} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{C}_{k+1}^r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \quad \text{and} \quad \boldsymbol{D}_{k+1} = \boldsymbol{1}, \tag{12}$$

where $\boldsymbol{1}$ is identity matrix and $\boldsymbol{0}$ is zero matrix. The matrices $\boldsymbol{A}_{k+1}^r$, $\boldsymbol{B}_{k+1}^r$ and $\boldsymbol{C}_{k+1}^r$ are matrices of the $k+1^{st}$ network with real number of ports and their sizes are $L_1 \times L_1$, $L_1 = n - m + 1$.

Substituting the relations (8) and (10) into (3), the relation (3) can be further written as follows

$$\left(\boldsymbol{Z}_{12}^k \cdot \boldsymbol{C}_{k+1}^r\right) \cdot \boldsymbol{U}_{2T,2}^{k+1} + \boldsymbol{U}_{2T,1}^{k+1} = \boldsymbol{U}_{2T,1}^k, \tag{13}$$

$$\left(\boldsymbol{A}_{k+1}^r + \boldsymbol{Z}_{22}^k \cdot \boldsymbol{C}_{k+1}^r\right) \cdot \boldsymbol{U}_{2T,2}^{k+1} = \boldsymbol{U}_{2T,2}^k, \tag{14}$$

$$\left(\boldsymbol{Z}_{32}^k \cdot \boldsymbol{C}_{k+1}^r\right) \cdot \boldsymbol{U}_{2T,2}^{k+1} + \boldsymbol{U}_{2T,3}^{k+1} = \boldsymbol{U}_{2T,3}^k. \tag{15}$$

The output voltage vector $\boldsymbol{U}_{2T,2}^k$ is the input voltage vector for the $k+1^{st}$ network and it is necessary to solve only relation (14) in order to obtain the output voltage vector

$$\boldsymbol{U}_{2T,2}^{k+1} = [\boldsymbol{A}_{k+1}^r + \boldsymbol{Z}_{22}^k \boldsymbol{C}_{k+1}^r]^{-1} \boldsymbol{U}_{2T,2}^k, \tag{16}$$

for the $k+1^{st}$ open ended network.

The impedance matrix of *ETS* for the $k+1^{st}$ network is

$$\boldsymbol{Z}_{2T,2}^{k+1} = [\boldsymbol{A}_{k+1}^r + \boldsymbol{Z}_{22}^k \boldsymbol{C}_{k+1}^r]^{-1} [\boldsymbol{B}_{k+1}^r + \boldsymbol{Z}_{22}^k \boldsymbol{D}_{k+1}^r]. \tag{17}$$

The last two recurrent relations are equal to the recurrent relations (3) and (4). The matrices used in these relations have the smaller dimension than the matrices used in the relations (3) and (4).

If all voltages

$$\boldsymbol{U}_{2T}^i = \left[\boldsymbol{U}_{2T,1}^i \mid \boldsymbol{U}_{2T,2}^i \mid \boldsymbol{U}_{2T,3}^i\right]^T, \tag{18}$$

where $i = k+1, k+2, ..., K$ and K is a total number of networks in cascade connection, are needed than the equation system (13-15) must be solved.

The solving procedure for cascade-connected networks with reduced number of ports is as follows:

1. The relations (3) and (4) are used to obtain $\boldsymbol{U}_{2T}^k$ and $\boldsymbol{Z}_{2T}^k$, i.e. *ETS* voltages and impedances for the first k cascade-connected networks. These vector and matrix are applied to the input ports of the next cascade-connected network.
2. The matrices $\boldsymbol{A}_{k+1}^r$, $\boldsymbol{B}_{k+1}^r$ and $\boldsymbol{C}_{k+1}^r$ are formed for the $k+1^{st}$ network with $2L_1$ ports.

3. At the junction between the k^{th} and $k+1^{st}$ networks, because of the reduced number of input ports, it is necessary to get only voltage vector $\boldsymbol{U}_{2T,2}^{k}$ from vector $\boldsymbol{U}_{2T}^{k}$ and matrix $\boldsymbol{Z}_{22}^{k}$ from matrix $\boldsymbol{Z}_{2T}^{k}$ (Eqs. (10) and (8)). The voltage vector $\boldsymbol{U}_{2T,2}^{k+1}$ and impedance matrix $\boldsymbol{Z}_{2T,2}^{k+1}$ are calculated from relations (16) and (17).
4. For the further calculation, $k+2, k+3, ..., K$, it is assumed $\boldsymbol{Z}_{2T}^{k+1} = \boldsymbol{Z}_{2T,2}^{k+1}$ and $\boldsymbol{U}_{2T}^{k+1} = \boldsymbol{U}_{2T,2}^{k+1}$ and the relations (3) and (4) can be used for solving the rest of the networks in the network cascade connection.

III. *ETS* VOLTAGE AND IMPEDANCE CALCULATION – A COMPLEX CASE

Solving procedure described in previously chapter can be also applied to complex network connections, as shown in Fig.3. The networks 1 and 2 are connected at the output ports of *ETS* with voltage $\boldsymbol{U}_{2T}^{k}$ and impedance $\boldsymbol{Z}_{2T}^{k}$. Number of ports for those networks are $2L_1$ and $2L_2$, respectively, where $L_1 = n - m + 1$ and $L_2 = q - p + 1$.

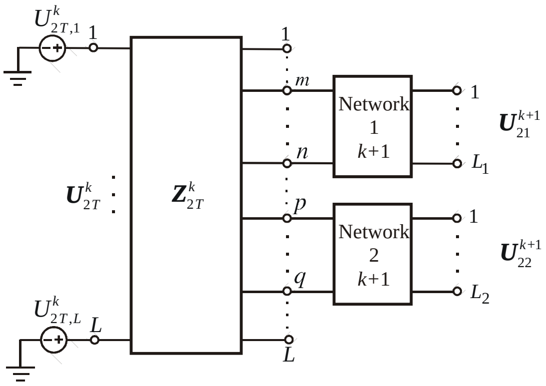

Fig. 3. Connection between the k^{th} network and the $k+1^{st}$ networks 1 and 2.

Solving procedure for this case is:

1. The voltage vector obtained by the equation (3) can be divided in the form

$$\boldsymbol{U}_{2T}^{k} = \left[\boldsymbol{U}_{2T,1m}^{k} \,|\, \boldsymbol{U}_{2T,mn}^{k} \,|\, \boldsymbol{U}_{2T,np}^{k} \,|\, \boldsymbol{U}_{2T,pq}^{k} \,|\, \boldsymbol{U}_{2T,qL}^{k}\right]^T, \quad (19)$$

where

$$\boldsymbol{U}_{2T,mn}^{k} = \left[U_{2T,m}^{k} \cdots U_{2T,n}^{k}\right]^T \quad (20)$$

and

$$\boldsymbol{U}_{2T,pq}^{k} = \left[U_{2T,p}^{k} \cdots U_{2T,q}^{k}\right]^T. \quad (21)$$

2. The next matrices are formed for both $k+1^{st}$ networks

$$\boldsymbol{A}_{k+1}^{r} = \begin{bmatrix} \boldsymbol{A}_{k+1}^{1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_{k+1}^{2} \end{bmatrix}, \; \boldsymbol{B}_{k+1}^{r} = \begin{bmatrix} \boldsymbol{B}_{k+1}^{1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B}_{k+1}^{2} \end{bmatrix} \text{ and}$$

$$\boldsymbol{C}_{k+1}^{r} = \begin{bmatrix} \boldsymbol{C}_{k+1}^{1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{C}_{k+1}^{2} \end{bmatrix}. \quad (22)$$

3. The impedance matrix given by (7) after the corresponding reduction is

$$\boldsymbol{Z}_{2T}^{k} = \begin{bmatrix} Z_{mm}^{k} & \cdots & Z_{mn}^{k} & | & Z_{mp}^{k} & \cdots & Z_{mq}^{k} \\ \vdots & \ddots & \vdots & | & \vdots & \ddots & \vdots \\ Z_{nm}^{k} & \cdots & Z_{nn}^{k} & | & Z_{np}^{k} & \cdots & Z_{nq}^{k} \\ - & - & - & | & - & - & - \\ Z_{pm}^{k} & \cdots & Z_{pn}^{k} & | & Z_{pp}^{k} & \cdots & Z_{pq}^{k} \\ \vdots & \ddots & \vdots & | & \vdots & \ddots & \vdots \\ Z_{qm}^{k} & \cdots & Z_{qn}^{k} & | & Z_{qp}^{k} & \cdots & Z_{qq}^{k} \end{bmatrix} = \begin{bmatrix} \boldsymbol{Z}_{11}^{k} & \boldsymbol{Z}_{12}^{k} \\ \boldsymbol{Z}_{21}^{k} & \boldsymbol{Z}_{22}^{k} \end{bmatrix} \quad (23)$$

4. The voltage vector $\boldsymbol{U}_{2T,2}^{k}$ in the relation (16) is

$$\boldsymbol{U}_{2T,2}^{k} = \left[U_{2T,mn}^{k} \,|\, U_{2T,pq}^{k}\right]^T \quad (24)$$

and impedance matrix $\boldsymbol{Z}_{22}^{k} = \boldsymbol{Z}_{2T}^{k}$, where $\boldsymbol{Z}_{2T}^{k}$ is defined with (23). The voltage vector

$$\boldsymbol{U}_{2T,2}^{k+1} = \left[U_{2T,mn}^{k+1} \,|\, U_{2T,pq}^{k+1}\right]^T \quad (25)$$

and $\boldsymbol{Z}_{2T,2}^{k+1}$ are calculated from the relations (16) and (17).
5. For the other networks in cascade connection till the outputs of networks 1 and 2 the relations (3) and (4) are used.

V. EXAMPLE

Consider a cascade-connected transmission lines on a $100\mu m$ GaAs substrate with $\varepsilon_r = 12.9$. Transmission lines are terminated in impedances $Z_S = 50\Omega$ at input ports and in impedances $Z_L = 150\Omega$ at output ports as shown in Fig.4. This system can be treated as impedance transformers with different number of sections in cascade connection. Two cases are observed:

Case I: Three cascade-connected microstip lines of equal lengths $d = 1000\mu m$ and different reduced widths $w1 = 500\mu m$, $w2 = 250\mu m$ and $w3 = 50\mu m$.

Case II: Six cascade-connected microstrip lines of equal lengths $d = 500\mu m$ and different reduced widths $w1 = 500\mu m$, $w2 = 400\mu m$, $w3 = 350\mu m$, $w4 = 250\mu m$, $w5 = 150\mu m$ and $w6 = 50\mu m$.

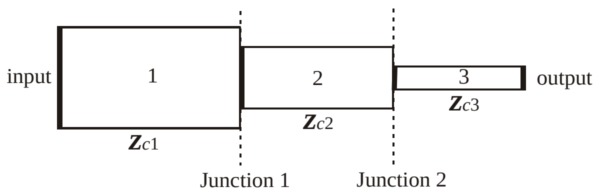

Fig. 4. Cascade connection of three microstrip lines of different reduced widths.

In both cases first line is terminated in impedances Z_S at input ports and the last line is terminated in impedances Z_L at output ports. The lines except the last one in both cases are analysed as open ended lines. For these lines the input voltages and impedances are corresponding *ETS* voltages $\boldsymbol{U}_{2T,2}$ and impedances $\boldsymbol{Z}_{2T,2}$. The voltages $\boldsymbol{U}_{2T,2}$ for the last line are real voltages at loads impedances Z_L. Lines are segmented arbitrary.

Scattering parameters of the sources and loads are

$$S_{11} = \frac{Z_{in} - Z_S}{Z_{in} + Z_S}, \qquad S_{22} = \frac{Z_L - Z_{out}}{Z_L + Z_{out}}, \tag{26}$$

where Z_{in} and Z_{out} are input and output impedances, respectively. These impedances are calculated as in reference [1]. Here, we observe magnitude of scattering parameters S_{11} and S_{22} for different number of sections in cascade. Figs. 5 and 6 show the magnitude of scattering parameters S_{11} and S_{22} versus frequency, respectively. Cascade connection of six microstrip lines gives better results in wider frequency band than cascade connection of three microstrip lines.

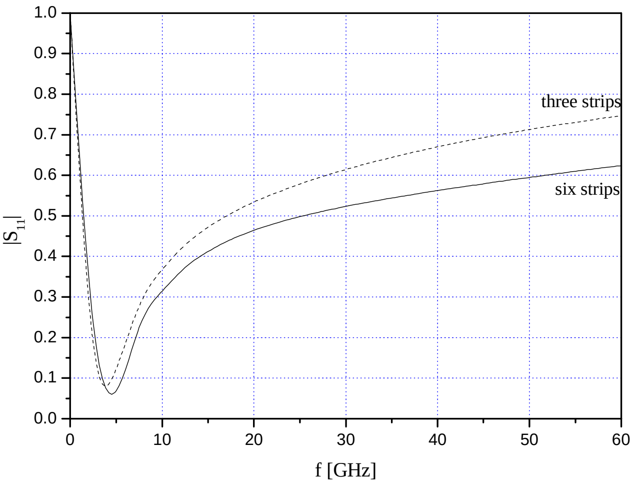

Fig. 5. Magnitude of the scattering parameter S_{11} versus frequency.

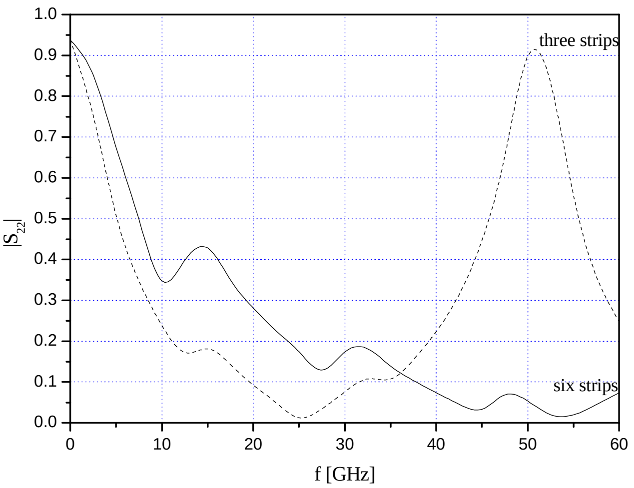

Fig. 6. Magnitude of the scattering parameter S_{22} versus frequency.

VI. Conclusion

ETS method given in the paper [1] is used for analysis of two-dimensional circuit represented as cascade-connected networks with equal number of input and output ports [3]. Here, it is shown that this method can be implemented for analysis of cascade-connected networks with different reduced number of ports. The additional relations needed for analysis of such two-dimensional circuits are evaluated. To verify the solving procedure, two examples of cascade-connected uniform microstrip lines are given.

It can be concluded that described analysis procedure can be applied to both symmetric and asymmetric connections of transmission lines with different lengths and reduced widths. The voltage vector and impedance matrix are reduced according to the junction, i.e. network connections.

The next step will be derivation of new algorithm for analysis some other types of various tapered microstrip transmission lines.

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