Various Aspects of Designing Filters by Model Order Reduction Techniques

Lj. Radic[1], W. Mathis[1]

***Abstract* - In this paper we will present properties of three different model order reduction methods for a control system all three applied on FIR filters: Balanced Model Order Reduction (BMR), Singular Perturbations Model Reduction (SPR) and Optimal Hankel-Norm Approximation (OHA). Also, we will discuss influences of group delay error of a reduced filter; pass band range, order and sampling frequency on level of reduction. In particular we will consider implementation of reduced filter. Our results from theoretical approaches will be illustrated by a concrete problem. We will show that the order of an FIR filter can be reduced by 78% preserving linear phase in pass band. The reduced filter is stable linear phase IIR filter, which is an essential part in our case of a complex audio system. It requires less power consumption than a higher order FIR filter.**

I. INTRODUCTION

The design of linear-phase digital filters has been considered in many publications because they allow distortion free transmission of signals. It is well known that adequate selection of the impulse response of a Finite Impulse Response (FIR) filter yields a linear phase characteristic, i. e. a constant group delay. The linear phase is one of the main advantages of FIR over IIR filters. However, the main drawback of FIR filters is their high order with respect to IIR filters, which have an equivalent magnitude response characteristic. A high filter dimensionality has two disadvantages: a long signal delay and high power consumption. The model order reduction techniques offer a solution for designing low order filter preserving amplitude and phase characteristics in the pass band of an original high order filter. These techniques can be grouped in two approaches: conventional and techniques converting FIR into IIR filters. Using the conventional technique an IIR filter is designed satisfying the magnitude specifications while ignoring the group delay. Then an all pass equalizer is designed and connected in cascade with the IIR filter to linearize the phase response [1], [2]. The three techniques, which we are going to discuss, are based on obtaining IIR filters from designed FIR filters, while simultaneously maintaining magnitude and phase characteristics in pass band [3].

Various techniques of model reduction, which belong to the second group, have been proposed in technical literature [4]-[6]. For reducing the order we revert to the known model reduction techniques such as Balanced Model Order Reduction (BMR) [4], Singular Perturbations Model Reduction (SPR) [5] and Optimal Hankel-Norm Approximation (OHA) [6]. These techniques are based upon obtaining a balanced realization of the original system, and then removing the weakly controllable and observable states [7]. All of these techniques transform high order FIR into low order IIR filters, which have the original magnitude response specification while maintaining a linear phase response in the pass band. Some of these techniques give better approximation of the FIR filter prototype at low frequencies, whereas others give better approximation at high frequencies. For the design of highly selective filters, it is therefore important that the error of approximation is small for all frequencies, especially in the transition band.

This paper is organized as follows. In the second section we briefly describe distinctive properties of all above mentioned reduction techniques. In some supplements, these different properties are characterized and evaluated by the magnitude error between reduced and original filter using various mathematical norms. Most representative norms are Hankel and L_∞ norm, which show whether two systems are close or far apart. Here, we present our conclusions derived from a mathematical approach to this two norms. The results that we achieved are exhibited also in the second section.

Furthermore, we discuss the influence of group delay error, extent of a pass band, order of an FIR filter and selected sampling frequency on the freedom of reduction in the third section.

In the section four we discuss implementation of reduced IIR filter. The main concern is the performance of the digital filter in finite word length implementation on one hand and the computational complexity of implementation on the other hand. Therefore, finite word length implementation of IIR filters can cause bad scaling, so that quantization noise is high. Further, high quantization noise leads to limit cycle oscillations. The scaling problem can be avoided by using low coefficient sensitivity filters [12]. These filters have very low quantization noise, so that they are limit cycle oscillation free filters. The structure that have a reasonably low coefficients sensitivity and a low round off noise level is parallel combination of two all-pass filters which sum is low pass and difference high pass filter. The decomposition of IIR filters into two all pass filters is one possible solution that we discuss.

As an interesting prototype application of model reduction techniques, we gave our suggested concept for digital class-D audio power amplifiers (ZePoC), showed in figure 1, that was developed in our laboratory [8], [9]. This concept includes a new coding algorithm such that audio signals can be encoded in binary signals. Because of precise output signals that were desired we used high order FIR filters for our digital implementation of ZePoC. These filters satisfied expectation, but from aspect of energy consumption and signal delay we

[1] Authors are with Electromagnetic Theory Group of the Institute of Electromagnetic Theory and Microwave Technique, University of Hannover, Appelstr. 9A, 30167 Hannover, Germany, Email: lrad@tet.uni-hannover.de

are looking for another possible solution using model order reduction techniques. The energy consumption was the main reason that forced us to a new solution using model reduction techniques.
In the section 5, we proof our theoretical results on ZePoC, and show the practical side of reduction methods.

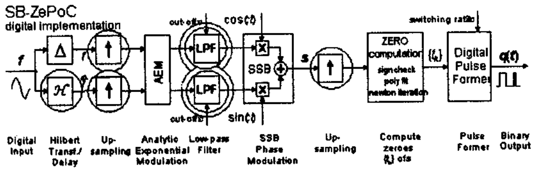

Fig.1: FIR filters of the ZePoC system are marked by circles. FIR filters with the highest order are designated by double circles.

II. Properties of the Model Reduction Techniques

Model reduction techniques, used so far for the approximation of an FIR by an IIR filter of reduced order, are based on the Singular Value Decomposition (SVD) approximation method [10]. This method is described by the singular values, which are important in deciding to what extent a given finite dimensional operator (order of a filter), can be approximated by one of the lower rank. This invariance can be attached to every constant finite-dimensional matrices and play the same role as the Hankel Singular Value (HSV) for dynamical systems. The HSVs determinate the complexity of the reduced system and provide an error bound for the resulting approximation. Moreover, the HSVs indicate those states components, which are weakly coupled to the input and the output of the dynamical system, in our case an FIR filter. The states that are recognized as "weak" can be discarded from the system, because of their neglecting influences. In other words, if the HSVs of an FIR filter are known the order of reduced IIR filter can be determined as the number of non-discarded HSVs. In these non-discarded HSVs the most significant properties of an FIR filter are contained and retained in the properties of an IIR filter. All model reduction techniques have as a starting point exactly the same HSVs, but using their own criterion they build reduced IIR filters with different transfer functions.
First method that we present is the Optimal Hankel-Norm Approximation (OHA). It is based on the dominant part of the HSVs, which are formed by the modified impulse response of the desired filter. The major objective of this design approach is to minimize the error between a reduced filter response and the desired one in the Hankel-norm sense [11]. The OHA method uses the HSVs of an FIR filter upon obtaining balanced realization for identification the global error bound for the resulting approximation. The balanced realization of a filter is related to two Lyapunov equations [12]

$$APA^T + BB^T = P, \qquad (1)$$

$$A^T QA + C^T C = Q, \qquad (2)$$

where the solutions P and Q are equal. These solutions are controllability and observability Grammians called, respectively. They depend on the matrices A, B and C, which describe relations between states, input signal and states, and output signal and states of the filter, respectively. This method sets aside HSV

$$\sigma_b := \sqrt{\lambda_b (P \cdot Q)}, \qquad (3)$$

which is determined as the first HSV that is larger than an error bound. $\lambda_g (P \cdot Q)$ is an equivalent eigenvalue of the matrix $P \cdot Q$. Indeed, the gain of the magnitude response error between the reduced filter response and the original one is equal to the first singular value that is less than selected σ_b.
Second method that we exhibit for order reduction is the Balanced Model Order Reduction (BMR). This method is based on equality of the controllability and observability Grammians, if the filter is realized in balanced state-space form. We use decomposition of the equalized controllability and observability Grammians into truncated and rejected parts. The truncated part contains HSVs that are larger than σ_b and represents controllability and observability Grammians that are reduced on truncated part. The rejected part contains all HSVs σ_i that satisfy $\sigma_i < \sigma_b$. These HSVs are related with the "weak" states.
Both described model reduction techniques can be compared using the Hankle and L_∞ norms. The norms of the reduced IIR filters are important, because they can be written in form of magnitude errors between an FIR filter and reduced IIR. The equations that expose magnitude errors of the OHA method in terms of Hankel and L_∞ norms are

$$\| G(z) - G_r(z)^{OHA} \|_H = \sigma_{k+1}, \qquad (4)$$

$$\| G(z) - G_r(z)^{OHA} \|_{L_\infty} \leq \sum_{i=k+1}^{N} \sigma_i. \qquad (5)$$

G(z) is the z-domain transfer function of the N^{th} order FIR filter and $G_r(z)^{OHA}$ is the z-domain transfer function of the reduced IIR filter that is obtained using the OHA method. σ_{k+1} is the first HSV that is smaller than selected global error bound and it satisfies $\sigma_{k+1} < \sigma_b$, where k is the order of reduced filter. The equation (4) exposes flat error magnitude of the OHA method with a gain equal to σ_{k+1}, with respect to the Hankel-norm. Moreover, the magnitude error obtained from L_∞ norm is bounded by the sum of all HSVs that are related to "weak" states and have to be rejected. In a similar way the Hankel and L_∞ norms for BMR method are defined

$$\| G(z) - G_r(z)^{BMR} \|_H \leq \sum_{i=k+1}^{N} \sigma_i , \qquad (6)$$

$$\| G(z) - G_r(z)^{BMR} \|_{L_\infty} \leq 2 \sum_{i=k+1}^{N} \sigma_i , \qquad (7)$$

where $G_r(z)^{BMR}$ is the z-domain transfer function of the reduced filter that is obtained using the BMR method. Equation (6) exposes the maximum absolute value of the frequency response, with respect to the Hankel-norm. For the same model order reduction method (BMR), equation (7) exposes upper limit of the magnitude error as the double sum of the HSVs related to the rejected states, with respect to the L_∞ norm. It is clear that the expected upper limit of magnitude error for the BMR method is higher than for the OHA method observing the L_∞ norm. Furthermore, it is expected that in most cases the upper limit of the Hankel norm for the OHA method is higher than the exact value of the same for the BMR method. However, experience has shown that these expectations are not always satisfied. Comparing the right sides of equations (4) and (6), we come up with conclusion that the OHA method provides a reduced filter with a smaller error than the BMR method only when the magnitude error $\| G(z) - G_r(z)^{BMR} \|_H$ belongs to the interval

$$(\sigma_{k+1}, 2 \sum_{i=k+1}^{N} \sigma_i] \qquad (8)$$

This is valid for $\sigma_{k+1} < 2 \sum_{i=k+1}^{N} \sigma_i$. Otherwise the BMR method can give a reduced filter with lower error. This problem is also discussed in [13], but from different approach and different conclusions are given, which do not exclude our results. Also, the obtained results in [13] are an integral part of our conclusion.

The last model reduction technique, which we used for approximation an FIR filter by IIR, was the Singular Perturbations Model Reduction (SPR). This method avoids computing the balancing transformation, which could run into numerical problems arising from singular matrices for very similar HSVs. A starting point for reduction is Lyapunov equation

$$AW_{CO}A + BC = W_{CO} , \qquad (9)$$

where W_{CO} is the cross-Grammian matrix that is defined as: $W_{CO} = \sum_{k=0}^{\infty} A^k BCA^k$. Further, the SPR method is based on determination the large and small (in magnitude) eigenvalues of the cross-Grammian matrix. These eigenvalues are significant for singular perturbational and truncated approaches, which are part of the SPR method. Indeed, the singular perturbation approach is used for the determination of a suitable decomposition of a system into weakly and strongly coupled subsystems, where the weakly coupled subsystem is eliminated.

For all mentioned model reduction techniques it is common that stability can be guaranteed if the full order model is stable. A more detailed discussion about stability of reduced system was shown by Silverman [14].

III. Influences of Different Parameters on Reduction

The quality of the magnitude and phase response of a reduced IIR filter can be evaluated by the magnitude response error and the group delay error. The magnitude response error we already discussed in the previous section. Further, we are going to discuss group delay error of a reduced filter, influences of pass band range, order and sampling frequency on level of reduction. Each of these parameters are observed when all other are fixed and do not have influence on reduction. For a concrete problem it is hard to isolate influence of one parameter because all mentioned parameters are correlated between each other so that influence of one can be annulled by the influence of the other one.

While the reduction methods maintain the magnitude specifications, each exhibits an approximately linear phase characteristic in pass band. As a measure of the phase linearity we use the group delay

$$\tau = -\frac{d\varphi}{dt} , \qquad (10)$$

where φ is phase of a filter. A linear phase filter satisfies

$$\varphi = const \Rightarrow \tau = 0; \qquad (11)$$

and

$$\Delta\tau = \tau_{\max} - \tau_{\min} = 0, \qquad (12)$$

where $\Delta\tau$ is group delay error, $\tau_{\min}$ and $\tau_{\max}$ are minimal and maximal group delay values. For an optimal reduced filter the approximately linear phase $\Delta\tau \approx 0$ is satisfied in pass band. Evidently, the value of $\Delta\tau$ gives a measure for phase linearity. If the value of $\Delta\tau$ is close to zero the phase is more linear. It was shown through many examples that the pass band group delay error of reduced filters, which are received by the OHA and the BMR methods, are independent of the frequency. For this reason they are not directly involved in discussions of relation between pass band extent and the level of reduction. However, for the SPR method ripple shape deviation ascends with frequency. This leads to the conclusion that the SPR method is less effective over significantly large frequency bands, because the phase linearity decreases with increasing frequency. Thus, for filter with large frequency bands we can expect better results from the other two methods.

Generally, a broader pass band is connected with a lower level of reduction whereas a smaller pass band leads to a higher reduction. The reason is that the linear phase is very important in pass band but not in stop band, so that a small pass band has additional degrees of freedom for reduction. In other words, everything that has minor influence on linear phase in pass band can be removed from original filter.

Further, the order of an FIR filter has significant influence on reduction. The reason lies in structure of reduced IIR filter. This filter has a parallel structure and two times more coefficients than its order is. It is very important for us that the number of coefficients is small, so that we would not have to implement more coefficients than for the FIR filter. This is possible only if the order of an FIR filter is reduced for at least 50%. Filters with very low order are hard to be reduced more than 50%, because degree of freedom for reduction is proportional to the order of a filter.

However, the reduction of FIR filter is not subordinated to a sampling frequency. We can assert that increasing the sampling frequency the order of an FIR filter with respect to desired specifications is also increased for the same factor k

$$f_N = kf_n \qquad k > 1, \tag{13}$$

$$N = kn = \frac{f_N}{f_n}, \tag{14}$$

where f_N and f_n are sampling frequencies of two FIR filters with order N and n. If the sampling frequency is increased for factor k, see equation (13), the number of HSVs is increased for $N-n$, because the number of HSVs and order of a FIR filter are the same. Further, the pass band has unchangeable range because pass- and stop band edges are steady. However, increasing sampling frequency stop band is increased. The pass band and previous stop band are described by existing HSVs, so that "new" HSVs describe broadened stop band and do not change previous N HSVs. They are smaller than σ_b and can be removed during reduction. This means that reduced IIR filters, obtained from FIR filters, with sampling frequencies f_N and f_n have same orders if all other properties are equal.

We will show using filter from ZePoC, figure 1, marked by double circles, that changing sampling frequency the order of the FIR filter is changed but not the order of the reduced IIR filter. The selected FIR filter has following specifications: stop band suppression of $-100dB$ and a cut-off frequency of $48kHz$. The sampling frequency varies between $200kHz$ and $800kHz$, what gives the order of the FIR filter between 88 and 352 for the same specifications, table 1. This filter is for each selected sampling frequency reduced on order 37. (We chose SPR method for reduction.) The group delay error $\Delta\tau$ is similar for all reduced filters and very small what gives linear phase in pass band. By figure 2 is presented how order N_{FIR} of the selected FIR and the reduced IIR filter N_{IIR} depend on sampling frequency f_s. The order of the FIR filter linear increases with sampling frequency but of the reduced IIR is constant.

TABLE I
SUMMARY OF ORDERS AND SAMPLING FREQUENCY OF THE FIR AND REDUCED IIR FILTERS.

$f_s[kHz]$	200	384	600	800
N_{FIR}	88	170	264	352
N_{IIR}	37	37	37	37
$\Delta\tau[ts]\times 10^{-6}$	1.088	0.594	1.203	1.577

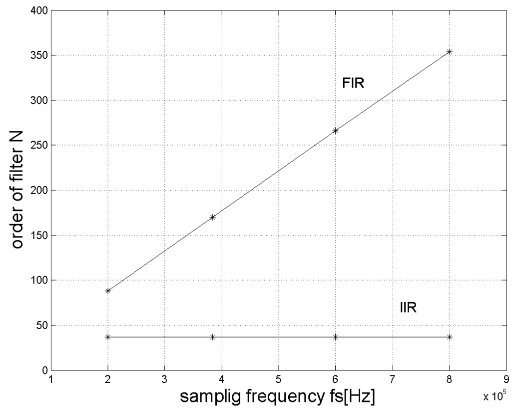

Fig. 2: Relation between order of the FIR and the reduced IIR filter and sampling frequency.

IV. LOW SENSITIVITY DIGITAL FILTER STRUCTURE

A digital filter implementation either by software or using hardware is different from its idealized design due to the available finite word-length for representing the multiplier coefficients. A low-sensitivity structure is very close to an ideal infinite-precision implementation. This structure is characterized by very small quantization noise related to quantized multiplier coefficients. Further, for these structures a parameter quantization can be chosen on that way that stability be guaranteed. One of many low-sensitivity structures is the all-pass section, shown in figure 3. This section exhibits very low pass band sensitivity and can be design to be free from parasitic oscillations [15], [17]. Also, the desired degree of phase linearity can be structurally incorporated. If one of the all-pass branches in the all-pass sum contains only delay elements, then the all-pass sum exhibits approximately linear phase in the pass band [17].

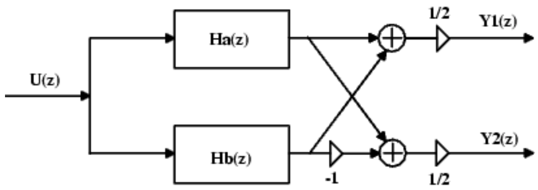

Figure 3: Low pass $Y_1(z)$ and high pass $Y_2(z)$ IIR filters presented as an all-pass sum. $H_a(z)$ and $H_b(z)$ are all-pass filters.

For reduced IIR filter implementation we use the property of low and high pass filters that those can be constructed using a parallel interconnection of two all-pass sections. For decomposition of low pass into two all pass filters, a complementary high pass filter is required, and other way around. In this case each all pass filter in branch has two times less order than the reduced one. Because of its parallel structure this decomposition gives much faster system with the same number of delay elements.

The question about linear phase filter presentation as an all-pass sum is opened. There are few proved theorems, which state that linear phase filters satisfy all conditions for this decomposition [18], [19]. The main property of transfer function of linear phase filters is symmetry or antisymmetry of its coefficients. The coefficients of any reduced IIR filter are not verbatim symmetric or antisymmetric, because of complex mathematical transformations that are applied to satisfied the phase linearity in pass band but not coefficients symmetry. However, reduced IIR filter has linear phase even the coefficients are not symmetric or antisymmetric. Thus, we can assert that if the transfer function of some filter has symmetric or antisymmetric coefficients than that filter has linear phase [16], but if some filter has linear phase it does not lead automatically to symmetric or antisymmetric coefficients. Because of this, the problem of implementation of reduced filter as low-sensitivity structure is not easy. For each separate case it is important reverting on basic low-sensitivity structures as that are lattice structures [17].

V. Example Designs

This section outlines a prototype FIR filter from the ZePoC system, as shown in figure 1, with required specifications and reduction of the resulting prototype to a lower order IIR filter using each of the presented techniques. The ZePoC system includes six filters, marked by circles in figure 1. As an example for reduction we choose the filter with the highest order and the following specifications: a pass band accuracy equivalent to 16 bits of resolution, a stop band suppression of $-100dB$, a cut-off frequency of $48kHz$ and a sampling rate of $384kHz$. The resulting FIR filter is of the order 170. For selected boundary error the HSV σ_b is linked with the order 37, see figure 4, what gives order of the reduced IIR filter. Described model reduction techniques reduce the FIR filter of order 170 into three IIR filters of order 37. The reduced filters have different magnitude and phase characteristics. We are going to compare the results of reduction, through magnitude response error and group delay error.

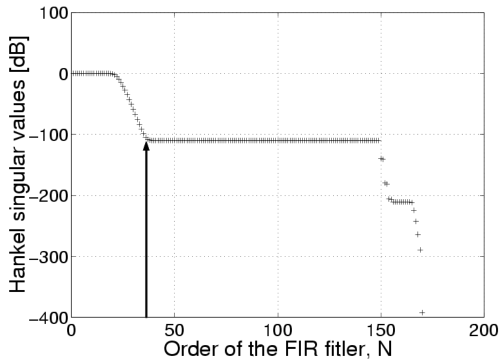

Fig. 4: Hankel singular values of FIR filter, $N = 170$.

The IIR filter, obtained using the SPR technique, has the smallest magnitude response error maintaining almost linear phase in pass band, see figures 5 and 6. Also, poles and zeros of the reduced filter are exhibited in figure 7. Positions of poles show that the reduced filter is stable. The general opinion that IIR filters cannot be stable and have a linear phase at the same time is inaccurate, especially if that filter is obtained from an FIR filter using reduction methods, [20].

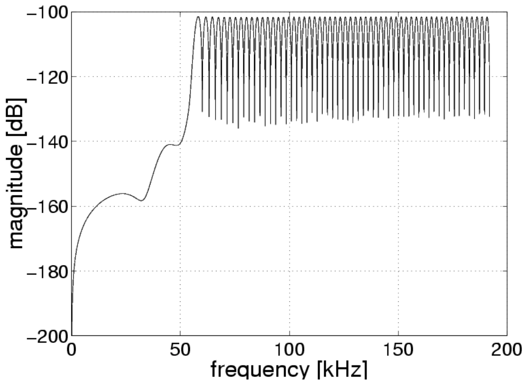

Figure 5: Magnitude response error, for the SPR method.

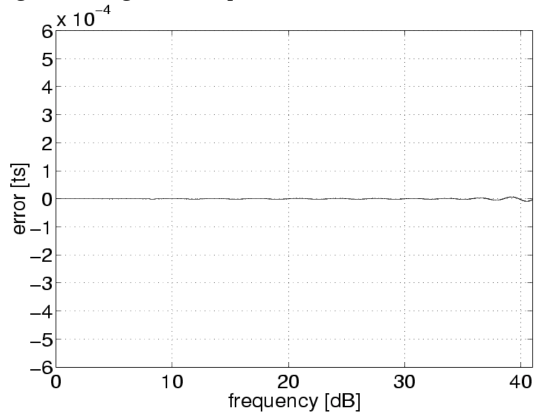

Figure 6: Pass-band group delay error shows existing ripple shape deviation from exact linearity in pass-band, for the SPR method.

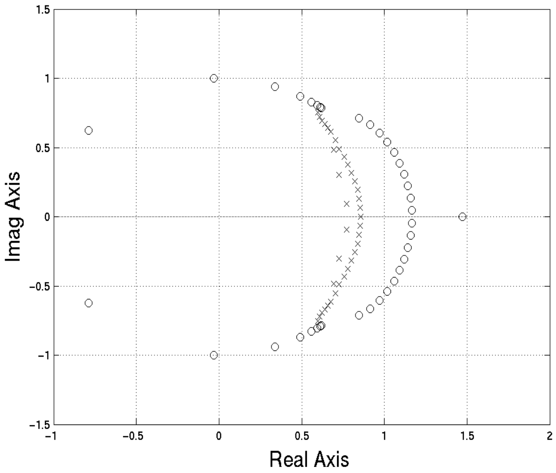

Figure 7: Position of poles and zeros in z plane presentation of the reduced IIR filter using the SPR method.

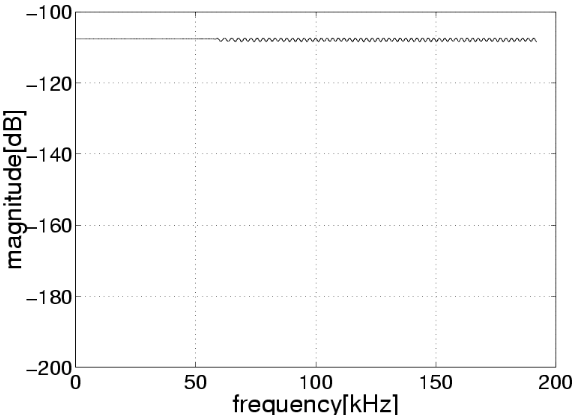

Figure 8: Magnitude response error, for the OHA method.

The results achieved by the other two methods are also interesting. As figure 8 shows, the error magnitude of the OHA is almost flat, with the gain around $-107dB$. Similar result is calculated by the Hankel-norm $\sigma_{k+1} \cong -107.66dB$, equation (4). Similarity of these two results confirms that only the "weak" states of the FIR filter were reduced. A better result is achieved by the BMR method, because a smaller magnitude error in the pass band is obtained, figure 9. For this method the magnitude error with respect to the Hankel and L_∞ norms do not reach the upper limit: $2\sum_{i=k+1}^{n} \sigma_i \cong -60.88dB$, equation (6). The magnitude error in the pass band is around $-137dB$; figure 9, what is out of range $(-107.66dB, -60.88dB]$ calculated as in (8). This result confirms the conclusion from the second section, where we discussed properties of different model reduction techniques.

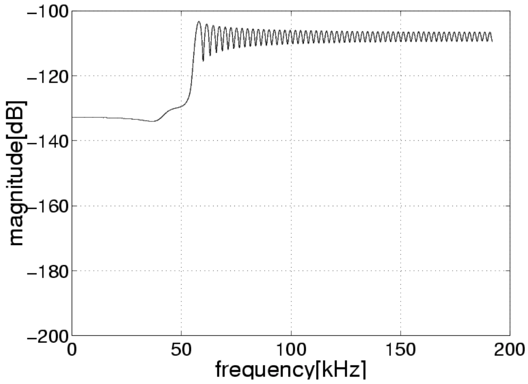

Figure 9: Magnitude response error, for the BMR method.

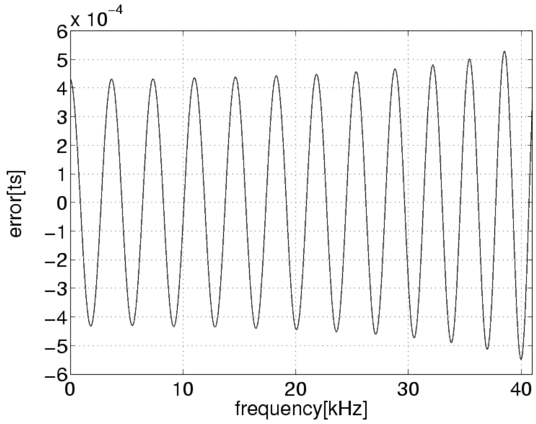

Figure 10: Pass-band group delay error shows existing ripple shape deviation from exact linearity in pass-band, for the OHA method.

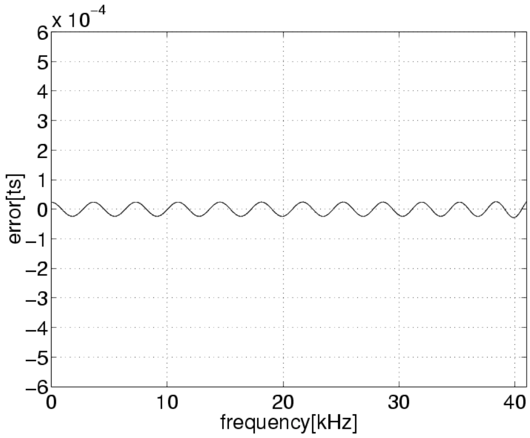

Figure 11: Pass-band group delay error shows existing ripple shape deviation from exact linearity in pass-band, for the BMR method.

Analyzing group delay error the SPR method shows a very small deviation of phase linearity, figure 6. This deviation ascends with frequency, but for the represented filter it is imperceptible, because of small pass band. The pass band group delay errors of the IIR filters, which are received by the OHA and the BMR methods, do not depend on frequency in the pass band, figures 10 and 11. The existing ripple shape deviation from exact linearity is bigger for these two methods than for the SPR. $\Delta\tau$ from equation (10) is closer to 0 for the SPR method than for OHA and BMR, so that the phase is more linear for the filter obtained by the SPR than by the other two methods, see figures 6, 10, 11. Also, this method does not lead to numerical problems, what is typical for the OHA method. The obtained IIR filter has linear phase, so that implementation as low-sensitive structure is easier than for filters with phase ripple [21].

VI. CONCLUSION

Through our discussion of reducing the order of an FIR digital filter, we expounded properties of the three standard model order reduction techniques. All three models order reduction methods transform high order FIR into low order IIR filters. The linear phase is preserved during the conversion from FIR into IIR filter. Further, we discussed the influence of different parameters group delay error; pass band range, order and sampling frequency on decreasing the order of an FIR filter. All presented model order reduction techniques meet the magnitude specifications of original FIR filter but differ in phase linearity. This is the reason for choosing the phase linearity as the reference for quality of reduction. We showed that the SPR method produces the smallest error in phase linearity; which increases with frequency. The phase linearity of the BMR and SPR methods is indifferent to frequency in the pass band. For this reason the SPR method is more applicable for filters with small pass-band than the two other methods. The SPR method also showed the best results for presented ZePoC system. However the hardware implementation of reduced IIR filter can be more complex than for the original FIR filter. In order to reduce design complexity, the reduced IIR filter can be decomposed in two all-pass sections. Hence, the signal delay can be reduced and coefficients with low sensitivity can be implemented, so that the resulting filter is a limit cycle oscillation free filter. Using all-pass sections an implementation of the reduced IIR filter can be facilitated.

REFERENCES

[1] C. Charalambous, A. Antoniou, "Equalization of recursive digital filters", *Proc. Inst. Elec. Eng*, vol 127, pt. G, no. 4, pp. 219-225, 1980.

[2] A. Antoniou, R. Howell, "Design of phase equalizers for recursive filters", *Proc. 1993 IEEE Pacific Rim*, pp. 104-107.

[3] G. Cortelazzo, M. R. Lightner, "Simultaneous design in both magnitude and group delay of IIR and FIR filters based on multiple criterion optimization", *IEEE Trans. Acous., Speech, Signal Processing*, pp. III-61-III-64, 1993.

[4] B. Beliczynski, I. Kale, G. D. Cain, "Approximation of FIR by IIR Digital Filters: An Algorithm Based on Balanced Model Reduction", *IEEE Trans. Signal Processing*, vol. 40, no. 3. Mar. 1992.

[5] R. W. Aldhaheri "Design of linear-phase IIR digital filters using singular perturbational model reduction", *IEE Proc.-Vsi. Image signal Process.*, vol. 147, no. 5, Oct. 2000.

[6] B. S. Chen, S. C. Peng, B. W. Chiou, "IIR filter design via optimal Hankel-norm approximation", *IEE, Proceedings-G*, vol. 139, no. 5, Oct. 1992.

[7] B. C. Moore, "Principal component analysis in linear systems: controllability, observability and model reduction", *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 17-31, 1981.

[8] M. Streitenberger, W. Mathis: "A Novel Coding Topology for Digital Class-D Audio Power Amplifiers with Very Low Pulse-Repetition Rate", *Proceedings of the* 28^{th} *European Solid-State Circuits Conference*, 24-26 Sept. 2002, Florence, Italy (ESSCIRC 2002), 515-518.

[9] D. Grabowski, M. Streitenberger, W. Mathis, "Model Reduction Techniques for the Design of Low-Power PLM Coders for Digital Audio Power Amplifiers", 16^{th} *European Conference on Circuit Theory and Design (ECCTD'03)*, 1-4 Sept. 2003, Krakow, Poland.

[10] W. H. Press, S. A. Teukolsky, W. T. Vetterling, B. P. Flannery, "Numerical Recipes in C", 2^{nd} *edition, Cambridge University Press*, Cambridge, U. K., 1992.

[11] K. Glover, "All optimal Hankel-norm approximations of linear multivariable systems and their L_∞-error bounds", *int. J. Control*, 1984, vol. 39, no. 6, pp. 1115-1193.

[12] B. C. Moore, "Principal component analysis in linear system: Controllability, observability, and model reduction", *IEEE Trans. Automat. Contr*., vol AC-26, pp. 17-32, Feb. 1981.

[13] I. Kale, J. Gryka, G. D. Cain, B. Beliczynski, "FIR filter order reduction: balanced model truncation and Hankel-norm optimal approximation", *IEE Proc.-Vis. Image Signal Process.*, Vol. 141, No. 3, pp. 168-174, June 1994.

[14] L. Prenobo, L. M. Silverman, "Model reduction via balanced state space representation", *IEEE Trans. Autom. Control*, April 1982, AC-27, (2), pp. 382-387.

[15] A. Fettweis, K. Meerkotter, "Suppression of parasitic oscillations in wave filters", *IEEE Trans. Circuits Syst*}, vol. CAS-22, pp. 239-246, Mar. 1975.

[16] F. Argenti, V. Cappellini, A. Sciorpes, A. N. Venetsanopoulos, "Design of IIR Linear-Phase QMF Banks Based on Complex Allpass Sections", *IEEE Transactions on Signal Processing*, vol. 44, no. 5, May 1996.

[17] A. H. Gray, Jr., "Passive cascaded lattice digital filters", *IEEE Trans. Circuits Syst*, vol. CAS-27, pp. 337-344, May 1980.

[18] P. P. Vaidyanathan, S. K. Mitra, Y. Neuvo, "A New Approach to the Realization of Low-Sensitivity IIR Digital Filters", *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP-34, no. 2, April 1986.

[19] T. Saramaki, "On the Design of Digital Filters as a Sum of Two All-Pass Filters", *IEEE Transactions on Circuits and Systems*, vol. CAS-32, no. 11, November 1985.

[20] S. K. Mitra, "Digital Signal Procesing", McGraw-Hill, 2^{nd} edition, New York, NY 10029, 2001.

[21] C. W. Kim, R. Ansari. "Approximately linear phase IIR filters using allpass sections", *Proc. IEEE Int. Symp. Circuits and Systems (ISCAS)*, vol. 2, pages 661-664, San Jose, May 5-7 1986.