

Structures Containing Left-Handed Metamaterials with Refractive Index Gradient: Exact Analytical Versus Numerical Treatment

M. Dalarsson¹, Z. Jakšić², P. Tassin³

Abstract – We investigated propagation of electromagnetic waves through composite structures with negative refractive index, the popular "left-handed metamaterials", for the case when there is a gradient of refractive index. We obtained the exact analytical solutions to the Helmholtz equation valid for arbitrary steepness of the graded interface between the positive and the negative index part. We analyzed the special case of matched impedances of the two constituent materials within the metamaterial composite. We derived analytical expressions for the field intensity, transmission and reflection coefficients and compared them with the results obtained by the numerical simulations using the Finite Element Method. The model allows for arbitrary spectral dispersion and lossy media.

Keywords – Electromagnetic Metamaterials, Left-Handed Metamaterials, Negative Refractive Index, Refractive Index Gradient.

I. INTRODUCTION

The negative refractive index metamaterials (NRM) [1], also known as left-handed metamaterials (LHM) are a new class of artificial composite materials. They are structures ordered at subwavelength level, furnishing a negative value of refractive index in a certain wavelength range. Typically they include negative magnetic permeability "particles" (for instance double split ring resonators) and negative dielectric permittivity ones (e.g. wire arrays, complementary split rings, etc.). NRM host a number of unusual properties. The direction of the Pointing vector in these materials is opposite to that of the wavevector, i.e. the vectors of the electric and magnetic field and the wavevector form a left-oriented set, contrary to conventional materials. The first theoretical consideration on the topic, written by Veselago, appeared as early as in [2]. In his seminal works Pendry [3], [4], [5] brought the concept to practical implementation, which resulted in a literal explosion of the interest for that topic ([1] and references therein). The first experimental confirmations were presented in [8]. Gradient index NRM could be useful in many various applications, for instance in lensing and filtering, for antireflection coatings, etc. Among especially important applications are superlenses [5], [6] and hyperlenses [7], which enable imaging below the diffraction limit through restoring evanescent waves in near field. Especially

interesting are hyperlenses, which convert near-field, subwavelength data to far field waves. Actually, any realistic structures containing both positive and negative index materials may well have a graded profile. Introduction of a refractive index gradient ensures an additional degree of freedom in their design.

Graded index NRM have been extensively studied. Ramakrishna described a metamaterial lens composed of gradient index media [9]. Smith et al [10] handled graded index metamaterials experimentally and proposed the use of such metamaterial lenses for the coupling with radiative elements in high-gain antenna applications. A numerical study of gradient index structures containing metamaterials was presented in [11]. Other papers on graded index NRM structures include [12], [13] and [14]. The determination of spectral parameters of metamaterial structures is currently mostly done by numerical simulation, and typically either by finite difference time domain method [15] or by finite elements method [16]. Analytical considerations for the case of graded negative refractive index materials have been discussed in [17], [18].

In this paper we solve analytically the Helmholtz equation for a structure with a graded interface between the positive and the negative refractive index region. We utilize the finite element method to numerically determine the electric field and the scattering parameters and to compare them with our analytical solution.

II. FIELD EQUATIONS

We start our analysis with the Maxwell equations and the only assumption we make at this point is that the fields are periodic in time, depending on $\exp(i\omega t)$ and that the material properties can be expressed by their effective dielectric permittivity and effective magnetic permeability. In the case of optical left-handed metamaterials this assumption is valid, since the nanostructuring to obtain negative refraction must be done at the subwavelength level. The geometry of the problem is illustrated in Figure 1.

The electric field vector is directed along the y axis, while the magnetic field vector is directed along the z-axis

$$\vec{E}(\vec{r}) = E_y \vec{y}_0 = E(x) \vec{y}_0 \quad , \quad \vec{H}(\vec{r}) = H_z \vec{z}_0 = H(x) \vec{z}_0 \quad , \quad (1)$$

¹M. Dalarsson is with Royal Institute of Technology, Falkv 7, 125 34 Alvsjo - Stockholm, Sweden, email: mardal@kth.se

²Z. Jakšić is with IHTM Institute of Microelectronic Technologies and Single Crystals, Belgrade, Serbia, email: jaksa@nanosys.ihtm.bg.ac.rs

³P. Tassin is with Vrije Universiteit Brussel, B-1050, Brussel, Belgium, email: philippe.tassin@vub.ac.be

so that the wave direction is along the x-axis. Since the fields are only dependent on the x-coordinate, we have

$$\frac{dE}{dx} = -i\omega\mu H \quad , \quad \frac{dH}{dx} = -i\omega\varepsilon E \quad , \quad (2)$$

where $\varepsilon = \varepsilon(\omega, x)$ and $\mu = \mu(\omega, x)$ are the frequency- and space-dependent dielectric permittivity and magnetic permeability, respectively.

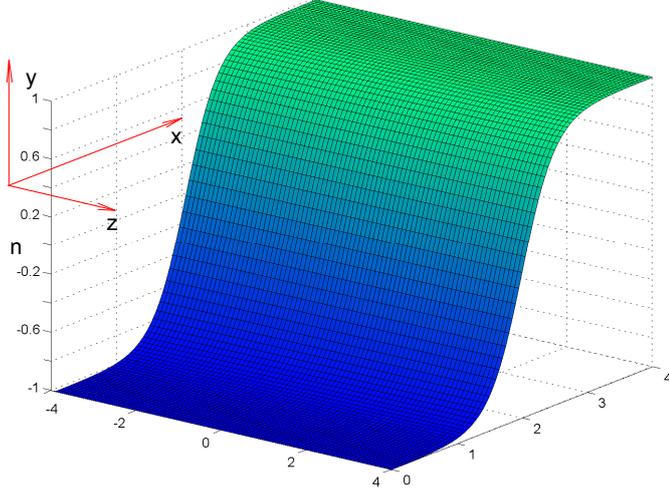


Fig. 1: Propagation of a perpendicular electromagnetic wave through a gradient index structure

From the equations (2) we obtain the two equations for the fields $E(x)$ and $H(x)$

$$\frac{d^2E}{dx^2} - \frac{1}{\mu} \frac{d\mu}{dx} \frac{dE}{dx} + \omega^2 \mu \varepsilon E = 0 \quad (3)$$

$$\frac{d^2H}{dx^2} - \frac{1}{\varepsilon} \frac{d\varepsilon}{dx} \frac{dH}{dx} + \omega^2 \mu \varepsilon H = 0 \quad (4)$$

III. SOLUTIONS OF THE FIELD EQUATIONS

The spectral properties and the gradual transition between the two materials is described by $\varepsilon = \varepsilon(\omega, x)$ and $\mu = \mu(\omega, x)$. The spatial dependence of these two functions can be expressed using various space functions, but we choose here the hyperbolic tangent function $\tanh x$ as the most convenient function, since it provides correct asymptotic values in both materials and allows a detailed study of the limit of the abrupt transition as well. We use the symmetric functions

$$\mu = -\mu_0 \mu_{eff}(\omega) \tanh(\rho x) \quad , \quad \varepsilon = -\varepsilon_0 \varepsilon_{eff}(\omega) \tanh(\rho x) \quad , \quad (5)$$

where ρ is an arbitrary parameter describing the abruptness of the transition from the right-handed material to the left of the plane $x = 0$ to the left-handed material to the right of the plane $x = 0$, and we assume that

$$\mu_{eff}(\omega) = \mu_R(\omega) \quad , \quad \varepsilon_{eff}(\omega) = \varepsilon_R(\omega) \left[1 + i \frac{\sigma(\omega)}{\omega \varepsilon_0 \varepsilon_R(\omega)} \right] \quad (6)$$

The dispersive functions $\varepsilon_R(\omega)$, $\sigma(\omega)$ and $\mu_R(\omega)$ are the relative permittivity, conductivity and relative permeability of the media, respectively.

The choice of the frequency dependence of these functions is fully arbitrary in the present analysis. One possible model is the Drude model, but there are no limitations in the choice of the model by the present approach. The impedance

$$Z = Z_0 Z(\omega) = \sqrt{\frac{\mu_0 \mu_{eff}(\omega)}{\varepsilon_0 \varepsilon_{eff}(\omega)}} \quad , \quad (7)$$

is constant throughout the entire observed space and there is no reflection at the interface between the two materials. An example of the two functions (3) with real $\varepsilon_{eff}(\omega)$ and $\mu_{eff}(\omega)$, is shown in Figure 2.

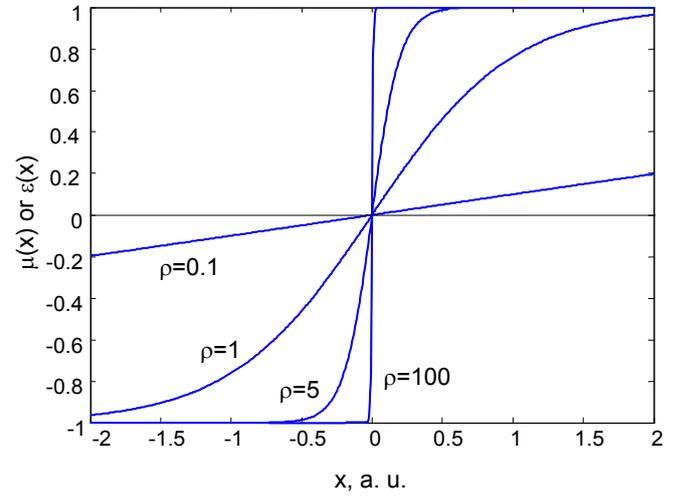


Fig. 2: Examples of the two functions μ_{eff} and ε_{eff} given by (5) for various values of ρ , ranging from near-abrupt ($\rho=100$) to slowly varying ($\rho=0.1$)

It turns out that, in this special case, the two differential equations (3) and (4) allow for a remarkably simple set of two independent exact solutions given by

$$E(x) = E_0 [\cosh(\rho x)]^{\pm i\kappa/\rho} \quad , \quad H(x) = H_0 [\cosh(\rho x)]^{\pm i\kappa/\rho} \quad , \quad (8)$$

where E_0 and H_0 are the amplitudes of the electric and magnetic fields of the incident electromagnetic wave far to the left ($x \rightarrow -\infty$) from the interface between the two materials, and

$$\kappa^2 = -\omega^2 \varepsilon_{eff}(\omega) \mu_{eff}(\omega) \quad , \quad \kappa = k + i\alpha \quad , \quad (9)$$

such that $k = \text{Re}(\kappa)$ and $\alpha = \text{Im}(\kappa)$.

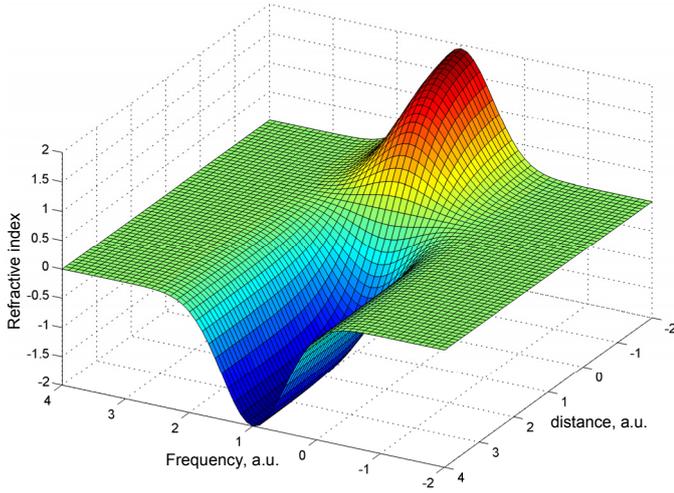


Fig. 3: Real part of refractive index versus frequency and distance across the RHM-LHM interface for a structure with both permittivity and permeability varying according to a hyperbolic tangent law and a Drude-type frequency dispersion.

The problem allows for arbitrary dispersion. Fig. 3 shows the dependence of the real part of the refractive index across the RHM-LHM interface for a structure with both permittivity and permeability varying according to a hyperbolic tangent law and a Drude-type frequency dispersion.

$$\varepsilon(\omega) = 1 - \frac{\omega_{pe}^2}{\omega(\omega + i\Gamma_e)} \quad , \quad (10)$$

$$\mu(\omega) = 1 - \frac{\omega_{pm}^2}{\omega(\omega + i\Gamma_m)} \quad , \quad (11)$$

thus

$$n_2(\omega) \approx 1 - \frac{\omega_p^2}{\omega^2} - i\Gamma \frac{\omega_p^2}{\omega^3} \quad , \quad (12)$$

where

$$\omega_{pe} = \omega_{pm} = \omega_p \quad , \quad (13)$$

is plasma frequency and

$$\Gamma_{pe} = \Gamma_{pm} = \Gamma_p \quad , \quad (14)$$

is the damping constant describing losses in material.

In the present problem we choose the solution with the minus sign in the exponent of the expressions (8), i.e.

$$E(x) = E_0 [\cosh(\rho x)]^{-i\kappa/\rho} \quad , \quad H(x) = H_0 [\cosh(\rho x)]^{-i\kappa/\rho} \quad , \quad (14)$$

For lossless media ($\sigma \rightarrow 0$) we obtain the asymptotic expressions for the fields $E(x)$ and $H(x)$ in the limits $x \rightarrow \pm \infty$, as

$$E(\mp\infty) = E_0 e^{i\kappa/\rho \ln^2} e^{\pm i\kappa x} \quad , \quad H(\mp\infty) = H_0 e^{i\kappa/\rho \ln^2} e^{\pm i\kappa x} \quad , \quad (15)$$

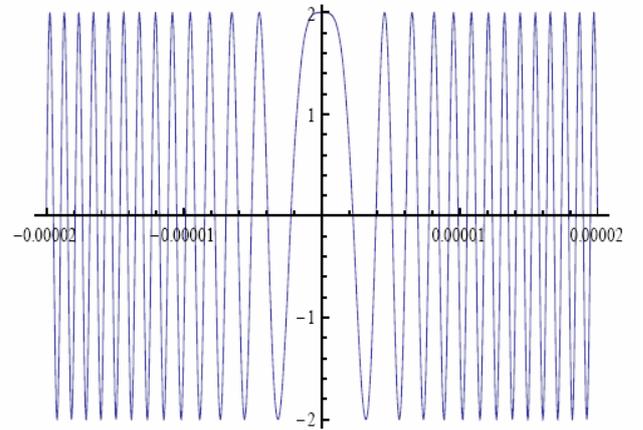
From the asymptotic expressions (15) we see that to the left of the interface at $x = 0$, i.e. in the right-handed material ($\varepsilon > 0$, $\mu > 0$), we have an electromagnetic wave with the wave vector

$$\vec{k}_1 = +k \vec{x}_0 \quad , \quad (16)$$

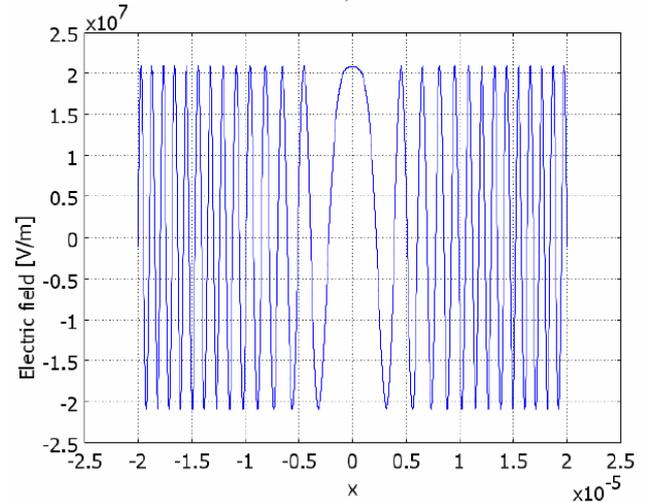
propagating to the right. On the other hand, to the right of the interface at $x = 0$, i.e. in the left-handed material ($\varepsilon < 0$, $\mu < 0$), we have an electromagnetic wave with the wave vector

$$\vec{k}_2 = -k \vec{x}_0 \quad , \quad (17)$$

propagating to the left without reflection.



a)



b)

Fig. 4: Comparison of the analytical (a) and numerical (b) results for the electric field

IV. NUMERICAL

The comparison of the exact analytic result (15) with the numerical calculation using Finite Element Method (FEM) simulation is shown in Fig. 4.

Figure 4 shows that there is an excellent agreement between the analytical calculations based on the result (15) and the numerical simulation using the Finite Element Method. Finally, Fig. 5 shows the analytical solution for electric field for $\rho = 10 \mu\text{m}^{-1}$ and for the incident angle $\theta = \pi/6$.

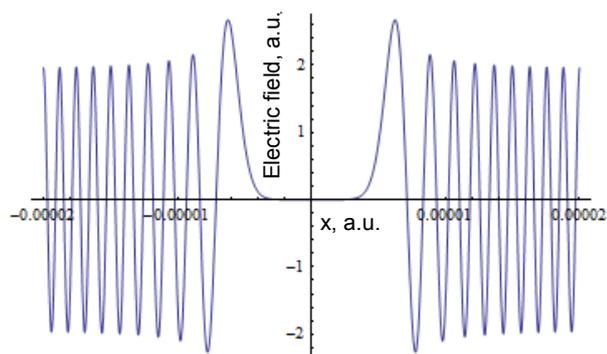


Fig. 5. Analytical solution for electric field $E(x, y)$ at $t = 0$ and $y = 0$; $\rho = 10 \mu\text{m}^{-1}$ and incident angle $\theta = \pi/6$.

V. CONCLUSION

We have investigated electromagnetic wave propagation across an interface between positive and negative refractive index material in case when there is a symmetric gradient of refractive index which can be described by hyperbolic tangent. We derived an exact analytic result for the electric field intensity and compared the analytical results with the corresponding results obtained by the numerical simulations using the Finite Element Method. We have shown that there is an excellent agreement between the analytical results and numerical simulations of the wave propagation through the interface between the two media. The model allows for arbitrary dispersion and the lossy media.

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