TLM Modelling of Left-Handed Metamaterials by Using Digital Filtering Techniques

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Abstract - **In this paper, a numerical three-dimensional (3D) model of electromagnetic left-handed metamaterials (LH MTM), is presented. The model is developed by using the Transmission Line Matrix (TLM) method based on Z-transforms, extended here to account for dispersive LH MTM properties in the time-domain. The dispersion function is based on the Drude model which allows the description of the electromagnetic properties of metamaterials over a wide frequency range. The accuracy, efficiency and stability of the proposed approach are verified using analytical solutions. In addition, some examples are shown which illustrate the unique characteristics of metamaterials.**

Keywords – **metamaterials**, **3D TLM method, bilinear Z transformation, dispersive time-domain model.**

I. Introduction

In essence, metamaterials (MTM) are periodic arrays of small metal and/or dielectric particles which in the bulk form, may be considered as artificial electromagnetic (EM) materials. When viewed in the bulk that is by calculating EM material properties using an averaging volume containing many particles, these materials may display extreme values of effective permittivity and permeability. For example, both the permittivity and the permeability may be negative, zero or unbounded in some frequency band. This promise of exotic EM properties has led to many types of MTM-based solutions being proposed for the realization of numerous types of microwave components having advanced characteristics and small size [1,2]. Of all types of responses these special media may exhibit, the negative refraction displayed by left-handed metamaterials (LH MTM) have attracted the most research attention in view of the potential applications in imaging and cloaking, etc.

For the purpose of investigations into the fundamental physics of LH MTM and to quantify the EM responses of practical structures for advanced component design, a number of differential and integral numerical techniques have been adapted to describe MTM properties both in the frequency-domain and the time-domain. For most problems of practical interest, these numerical techniques typically offer much faster analyses than the implementation of the MTM transmission line networks using circuit simulators. The most popular differential numerical techniques in the time-domain are the Finite-Difference Time-Domain (FD-TD) method [3] and the Transmission-Line Matrix (TLM) method [4]. Incorporation of MTM properties into these approaches allows the time-harmonic and transient simulation of MTM structures for direct analysis of their dispersive behaviour.

The FDTD method is widely used for modeling EM wave interaction with complex materials and several techniques have been developed to incorporate the frequency dispersion of MTM. Some of these techniques are detailed and referenced in [3]. Closely related to FDTD, the TLM method is a numerical network model of Maxwell's field equations. Because it is a network model, TLM is therefore perfectly suited to description of MTM as host transmission lines with embedded lumped series capacitors and shunt inductors [2]. Following this loaded transmission line approach for the description of MTM, the TLM method has been enhanced in [5] using a model based on the insertion of reactive elements between the nodes of the conventional TLM mesh of link lines. However, in contrast to the enhanced FDTD method for MTM modelling, the TLM model of [5] is of limited applicability because it only allows the specification of lossless MTM properties at one particular design frequency, i.e. that frequency for which the embedded reactive stub values were calculated.

In order to capture the physics of the time-dependent dispersive behavior of LH MTM, it is necessary to devise a scheme which is valid over a wide frequency range. One variation of the TLM method which is naturally suited to the description of arbitrary time-dependent responses is based on Z-transforms [6]. This approach has been successful in the development of numerical schemes for the time-domain treatment of general frequency-dependent properties in isotropic, bi-isotropic, anisotropic and nonlinear materials that can be found in nature [6,7]. In this paper, the Z-transform TLM technique is extended for the time-domain description of LH MTM responses. The Drude model is adopted to describe the frequency-dependent properties of the MTM which may be specified either by the permittivity and permeability or the electric and magnetic conductivities. The bilinear Z-transform is then used to transfer this dependence in the discrete time-domain and so develop the numerical procedure for incorporation into the three-dimensional (3D) TLM mesh. The technique is applied to three canonical examples. Firstly it is used to simulate an example which highlights some of the unique characteristics of LH MTM (i.e. impedance matching and negative phase velocity). The second example involves the calculation of the reflection coefficient of an air-MTM interface to validate and verify the

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accuracy of the proposed model over a wide frequency range. In the final example, the approach is applied to the simulation of EM wave interaction with a lossless or lossy LH MTM interface having a graded refractive index profile. All the numerical calculations yield results which confirm either expected MTM behaviour or yield close agreements with analytical solutions.

II. Dispersive 3D TLM Z-Transform Model of Metamaterials

The details of the Z-transform based TLM approach for the simulation of various types of conventional linear time-dependent materials can be found in [6,7]. As described in those papers, the main task for the description of the time-dependence is to determine by using an appropriate susceptibility or conductivity model the elements of the partial fraction expansions forms shown below:

$$(1-z^{-1})\chi_{e,m}(z) = \chi_{e,m0} - z^{-1}(\chi_{e,m1} + \overline{\chi_{e,m}(z)}) \quad (1)$$

$$(1+z^{-1})g_e(z) = g_{e0} + z^{-1}(g_{e1} + \overline{g_e(z)}) \quad (2)$$

$$(1+z^{-1})r_m(z) = r_{m0} + z^{-1}(r_{m1} + \overline{r_m(z)}) \quad (3)$$

In these expressions, χ_e is the electric susceptibility, χ_m is the magnetic susceptibility, g_e is the normalized electric conductivity and r_m is the normalized magnetic resistivity. Depending on the bandwidth of interest, typical realistic LH MTM responses can be characterized by using either the Drude or Lorentz dispersion models. These models are different in that the real parts of the MTM properties are negative in different regions of the frequency band. In this paper, the Drude model is used for both the susceptibility and the conductivity models. Both models give identical results. The Drude models are:

$$\varepsilon(\omega) = \varepsilon_0\left(\varepsilon_\infty - \frac{\omega_{pe}^2}{\omega^2 - j\omega\gamma_e}\right) \quad (4)$$

$$\mu(\omega) = \mu_0\left(\mu_\infty - \frac{\omega_{pm}^2}{\omega^2 - j\omega\gamma_m}\right) \quad (5)$$

$$\sigma_e(\omega) = \frac{\sigma_{e0}}{1 + j\omega\tau_e} \quad (6)$$

$$\sigma_m(\omega) = \frac{\sigma_{m0}}{1 + j\omega\tau_m} \quad (7)$$

In these expressions, $\omega_{pe,m}$, $\gamma_{e,m}$ and $\sigma_{e,m0}$ are the electric or magnetic plasma frequencies and the corresponding collision frequencies and static conductivities, respectively. Electric and magnetic collision times can be expressed through corresponding collision frequencies as $\tau_{e,m} = 1/\gamma_{e,m}$. For a LH MTM which is matched to free-space, the static electric and magnetic conductivities are related by $\sigma_{m0} = \eta_0^2\sigma_{e0}$ where η_0 is the wave impedance of free-space.

As it can seen from Eqs. (4)-(7), identical dispersion forms are assumed for both the dielectric and magnetic properties. For brevity and clarity of the following formulation, only the *y*-component of the electric field and the *z*-component of the magnetic field are considered in this paper. The other field components are obtained using similar expressions. Details of the inclusion of the Z-transform models into the 3D TLM method are outlined in [6,7].

A. Susceptibility Model

Using the notation of [6], the electric and magnetic susceptibilities of the LH MTM in the s-domain described by Eqs. (4) and (5) are:

$$\chi_e(s) = \chi_{e\infty} + \frac{\omega_{pe}^2}{\gamma_e}\left(\frac{1}{s} - \frac{1}{s+\gamma_e}\right) \quad (8)$$

$$\chi_m(s) = \chi_{m\infty} + \frac{\omega_{pm}^2}{\gamma_m}\left(\frac{1}{s} - \frac{1}{s+\gamma_m}\right) \quad (9)$$

In the next step of the algorithm development, the bilinear Z-transform s→2(1-z[-1])/[Δ*t*(1+z[-1])] is applied to Eqs. (8) and (9) instead of the exponential Z-transform adopted in [6]. This is because the bilinear Z-transform produces a much more accurate numerical scheme, especially at high frequencies. The electric and magnetic susceptibilities in the Z-domain are:

$$\chi_e(z) = \chi_{e\infty} - K_e(1+z^{-1})\left(\frac{1}{1-z^{-1}} - \frac{1}{B_{se}(1-z^{-1}A_{se}/B_{se})}\right) \quad (10)$$

$$\chi_m(z) = \chi_{m\infty} - K_m(1+z^{-1})\left(\frac{1}{1-z^{-1}} - \frac{1}{B_{sm}(1-z^{-1}A_{sm}/B_{sm})}\right) \quad (11)$$

where the coefficients are: $K_{e,m} = -\omega_{pe,m}^2\Delta t/(2\gamma_{e,m})$, $B_{se,m} = 1+\gamma_{e,m}\Delta t/2$ and $A_{se,m} = 1-\gamma_{e,m}\Delta t/2$.

As described in [6,7], by taking partial fraction expansions an explicit algorithm may be devised. This is because the time-dependent part of the electric and magnetic susceptibility functions are updated using the values of the electric and magnetic field value saved from the previous time-step:

$$(1-z^{-1})\chi_e(z) = \chi_{e\infty} - K_e + K_e/B_{se} - z^{-1}\left(\chi_{e\infty} + K_e - a_{se}K_e/B_{se} + z^{-1}\frac{b_{se}/4}{1-z^{-1}a_{se}}\right) \quad (12)$$

$$(1-z^{-1})\chi_m(z) = \chi_{m\infty} - K_m + K_m/B_{sm} - z^{-1}\left(\chi_{m\infty} + K_m - a_{sm}K_m/B_{sm} + z^{-1}\frac{b_{sm}/4}{1-z^{-1}a_{sm}}\right) \quad (13)$$

Comparing these expressions with Eq. (1), the elements of partial fraction expansions forms are identified, i.e.:

$$\chi_{e,m0} = \chi_{e,m\infty} - K_{e,m} + K_{e,m}/B_{se,m} \quad (14)$$

$$\chi_{e,m1} = \chi_{e,m\infty} + K_{e,m} - a_{se,m}K_{e,m}/B_{se,m} \quad (15)$$

$$\overline{\chi_{e,m}(z)} = z^{-1}\frac{b_{se,m}/4}{1-z^{-1}a_{se,m}} \tag{16}$$

where the coefficients are: $a_{se,m} = A_{se,m}/B_{se,m}$ and $b_{se,m}/4 = K_{e,m}4\gamma_{e,m}\Delta t/[2B_{se,m}{}^3]$.

In summary, the update scheme for the y-component of the electric field and the z-component of the magnetic field requires the additional accumulators S_{se} and S_{sm}, respectively. These accumulators are evaluated by the definition of the state-variables $X_{se} = z^{-1}V_y/(1-z^{-1}a_{se})$ and $X_{sm} = z^{-1}i_z/(1-z^{-1}a_{sm})$:

$$S_{se} = 4\overline{\chi_e(z)}V_y = b_{se}X_{se} \tag{17}$$

$$S_{sm} = 4\overline{\chi_m(z)}i_z = b_{sm}X_{sm} \tag{18}$$

The 3D TLM scheme involves the electric and magnetic field calculations and the update of the accumulators and the state-variables. Including a frequency-independent normalized electric conductivity term g_e and a normalized magnetic resistivity term r_m, the scattering process is:

$$V_y = T_{se}(2V_y^r + z^{-1}S_{ey}), \quad i_z = T_{sm}(-2i_z^r + z^{-1}S_{mz}) \tag{19}$$

$$S_{ey} = 2V_y^r + k_{se}V_y + b_{se}X_{se}, \quad S_{mz} = -2i_z^r + k_{sm}i_z + b_{sm}X_{sm} \tag{20}$$

$$X_{se} = z^{-1}a_{se}X_{se} + z^{-1}V_y, \quad X_{sm} = z^{-1}a_{sm}X_{sm} + z^{-1}i_z \tag{21}$$

with the coefficients: $T_{se} = (4 + g_e + 4\chi_{e0})^{-1}$, $T_{sm} = (4 + r_m + 4\chi_{m0})^{-1}$, $k_{se} = -(4 + g_e - 4\chi_{e1})$ and $k_{sm} = -(4 + r_m - 4\chi_{m1})$.

B. Conductivity Model

The dielectric and magnetic conductivities of the LH MTM in the s-domain can also be expressed by using the normalized conductivity and resistivity as shown in [6,7]:

$$g_e(s) = \sigma_e(s)\eta_0\Delta l = \frac{g_{ec}}{1+s\tau_e} \tag{22}$$

$$r_m(s) = \sigma_m(s)\Delta l/\eta_0 = \frac{r_{mc}}{1+s\tau_m} \tag{23}$$

where $g_{ec} = \sigma_{e0}\eta_0\Delta l$ and $r_{mc} = \sigma_{m0}\Delta l/\eta_0$.

Applying the bilinear Z-transform $s\rightarrow 2(1-z^{-1})/[dt(1+z^{-1})]$ to Eqs. (22) and (23), the following representation in the Z-domain is obtained:

$$g_e(z) = (1+z^{-1})\frac{g_{ec}}{B_{ce}(1-z^{-1}A_{ce}/B_{ce})} \tag{24}$$

$$r_m(z) = (1+z^{-1})\frac{r_{mc}}{B_{cm}(1-z^{-1}A_{cm}/B_{cm})} \tag{25}$$

where the coefficients are: $A_{ce,m} = 2\tau_{e,m}/\Delta t - 1$ and $B_{ce,m} = 2\tau_{e,m}/\Delta t + 1$.

The numerical model follows from the partial fraction expressions of Eqs. (2) and (3) leading to:

$$(1+z^{-1})g_e(z) = \frac{g_{ec}}{B_{ce}} + z^{-1}\left[\frac{g_{ec}(2+A_{ce}/B_{ce})}{B_{ce}} + z^{-1}\frac{g_{ec}(1+2A_{ce}/B_{ce}+A_{ce}^2/B_{ce}^2)}{B_{ce}(1-z^{-1}A_{ce}/B_{ce})}\right] \tag{26}$$

$$(1+z^{-1})r_m(z) = \frac{r_{mc}}{B_{cm}} + z^{-1}\left[\frac{r_{mc}(2+A_{cm}/B_{cm})}{B_{cm}} + z^{-1}\frac{r_{mc}(1+2A_{cm}/B_{cm}+A_{cm}^2/B_{cm}^2)}{B_{cm}(1-z^{-1}A_{cm}/B_{cm})}\right] \tag{27}$$

The elements of the partial fraction expansions are:

$$g_{e0} = \frac{g_{ec}}{B_{ce}} \tag{28}$$

$$g_{e1} = \frac{g_{ec}}{B_{ce}}(2 + A_{ce}/B_{ce}) = g_{e0}(2 + a_{ce}) \tag{29}$$

$$\overline{g_e(z)} = z^{-1}\frac{g_{ec}(1+2A_{ce}/B_{ce}+A_{ce}^2/B_{ce}^2)}{B_{ce}(1-z^{-1}A_{ce}/B_{ce})} = \frac{z^{-1}b_{ce}}{1-z^{-1}a_{ce}} \tag{30}$$

$$r_{m0} = \frac{r_{mc}}{B_{mc}} \tag{31}$$

$$r_{m1} = \frac{r_{mc}}{B_{cm}}(2 + A_{cm}/B_{cm}) = r_{m0}(2 + a_{cm}) \tag{32}$$

$$\overline{r_m(z)} = z^{-1}\frac{r_{mc}(1+2A_{cm}/B_{cm}+A_{cm}^2/B_{cm}^2)}{B_{cm}(1-z^{-1}A_{cm}/B_{cm})} = \frac{z^{-1}b_{cm}}{1-z^{-1}a_{cm}} \tag{33}$$

where the coefficients are: $a_{ce,m} = A_{ce,m}/B_{ce,m}$, $b_{ce} = g_{e0}(1 + 2a_{ce} + a_{ce}^2)$ and $b_{cm} = r_{m0}(1 + 2a_{cm} + b_{cm}^2)$.

As in the case of the susceptibility model, additional conductivity material accumulators S_{ce} and S_{cm} are required:

$$S_{ce} = \overline{g_e(z)}V_y = b_{ce}X_{ce} \tag{34}$$

$$S_{cm} = \overline{r_m(z)}i_z = b_{cm}X_{cm} \tag{35}$$

where: $X_{ce} = z^{-1}V_y/(1-z^{-1}a_{ce})$ and $X_{cm} = z^{-1}i_z/(1-z^{-1}a_{cm})$.

The 3D TLM scattering process can be then described as follows:

$$V_y = T_{ce}(2V_y^r + z^{-1}S_{ey}), \quad i_z = T_{cm}(-2i_z^r + z^{-1}S_{mz}) \tag{36}$$

$$S_{ey} = 2V_y^r + k_{ce}V_y - b_{cm}X_{ce}, \quad S_{mz} = -2i_z^r + k_{cm}i_z - b_{cm}X_{cm} \tag{37}$$

$$X_{ce} = z^{-1}a_{ce}X_{ce} + z^{-1}V_y, \quad X_{cm} = z^{-1}a_{cm}X_{cm} + z^{-1}i_z \tag{38}$$

with the coefficients: $T_{ce} = (4 + g_{e0} + 4\chi_{e\infty})^{-1}$, $T_{cm} = (4 + r_{m0} + 4\chi_{m\infty})^{-1}$, $k_{ce} = -(4 + g_{e1} - 4\chi_{e\infty})$ and $k_{cm} = -(4 + r_{m1} - 4\chi_{m\infty})$.

III. Model Verification

A. Metamaterial Slab

In this first example, the behaviour of EM waves interacting with a MTM slab is investigated. The geometry of the problem is shown in Fig. 1.

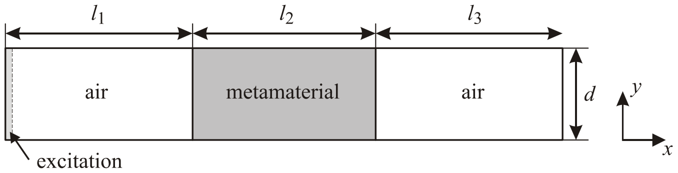

Fig. 1. Geometry of the metamaterial slab surrounded by air

The length of each region was l_1= l_2= l_3=70 mm and the width was d=10 mm. The parameters of the Drude function were chosen so that at 10GHz, the refractive index of the LH MTM was n=-2 (i.e. ε_r = μ_r = -2 at 10GHz) and its characteristic impedance was η_0 (i.e. approximately 377 Ω). The losses in the MTM slab were very small and therefore may be neglected (i.e. $\omega_{pe,m} >> \gamma_{e,m}$). The excitation was a monochromatic 10GHz TEM wave with its electric field polarized in z-direction. For the modelling of this two-dimensional structure, a uniform TLM mesh with 210×10×1 nodes was used. Electric and magnetic boundary conditions were applied on the z- and y-walls, respectively, in order to support TEM wave propagation. The electric field distributions at times t_1 and t_2, separated by 10Δt interval, are shown in Fig. 2. From this figure, it can be confirmed that the wave impedance of the LH MTM slab is matched to air, the phase velocity is negative inside the MTM slab and that the wavelength in the MTM is half the wavelength in air.

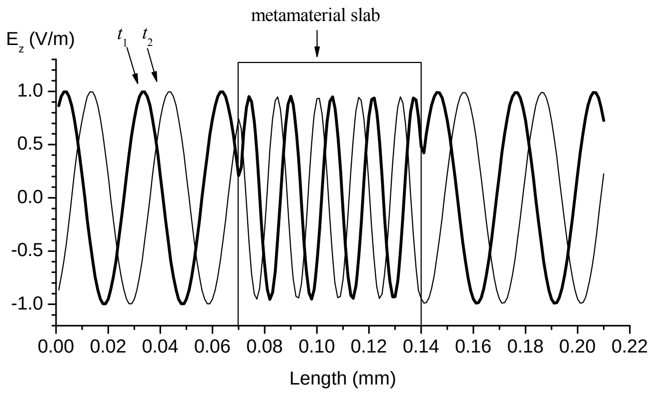

Fig. 2. Electric field distribution, $t_2 - t_1$=10Δt

B. Metamaterial Interface

The capability of the proposed 3D TLM model to account for the frequency dispersive behaviour of LH MTM is illustrated by the calculation of the reflection coefficient of an air-MTM interface over a wide frequency range. The parameters of the Drude function are chosen differently for permittivity and permeability of LH MTM so that in this example, ε_r = -2 and μ_r = -1 at 10GHz. From elementary analysis, it may be shown that for this selection of material properties, the reflection coefficient magnitude at 10GHz is 0.172. The frequency dependence of the real part of the relative permittivity and permeability of this LH MTM is shown in Fig. 3.

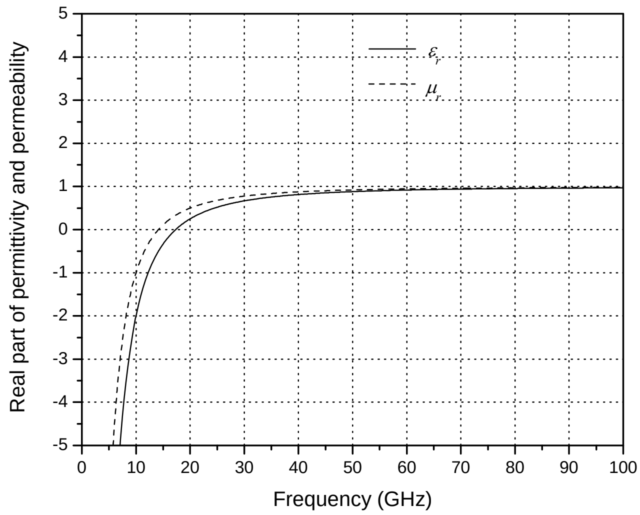

Fig. 3. Real part of relative permittivity and permeability of LH MTM versus frequency

An initial pulse of Gaussian form was used to introduce energy into the simulation over a broad frequency range. The numerical procedure to calculate the wide bandwidth reflection coefficient of the air-MTM interface was performed using two TLM simulations: In the first simulation, the incident field was calculated. For this, the entire problem space was filled with air and the time-dependent electric field at the observation point was recorded. In the second simulation the electric field was recorded with the MTM slab in place. The result of the second simulation provided the total field at the observation point one node in front of the interface. The reflected field was obtained by subtracting the incident field from the total field. The incident and reflected electric fields versus time are shown in Fig. 4.

The incident and reflected electric fields in the time-domain were transformed to the frequency-domain using the discrete Fourier transform. The reflection coefficient magnitude at each frequency was calculated by dividing the transform of the reflected field by the transform of the incident field. Fig. 5 shows the close agreement between the reflection coefficient magnitude obtained using the TLM simulation and an analytical model.

C. Graded Metamaterial Interface

In the final example, the TLM Z-transform model is used to investigate EM wave propagation across an interface between positive and negative refractive index materials for the case where there is a symmetric gradient of refractive index. This gradient is taken to be a hyperbolic tangent (tanh) function. This form is convenient as it provides the correct asymptotic values in both materials and there is an analytical solution of this problem for either lossless case [8] or arbitrary losses [9]. Because the solution allows for a fully arbitrary choice of the frequency-dependence of the EM properties, it was used to validate the presented TLM model at two different frequencies. For the lossless case, the interface was illuminated with a plane wave having a frequency of 30GHz (i.e. λ=1 cm), while for the lossy interface plane wave of 300THz (i.e. λ=1 μm) was used assuming that imaginary part of relative permittivity and permeability is 0.01 at that frequency. In both cases, the tanh function shown in Fig. 6, is chosen to represent variation of real parts of permittivity and permeability on the interface at considered frequencies. The electric field observed near the lossless and lossy interface is shown in Fig. 7 and 8, respectively. These results show close agreement between the analytical and TLM Z-transform model results. It should be said that in all examples considered in this section, both the susceptibility and conductivity TLM Z-Transform models gave identical results.

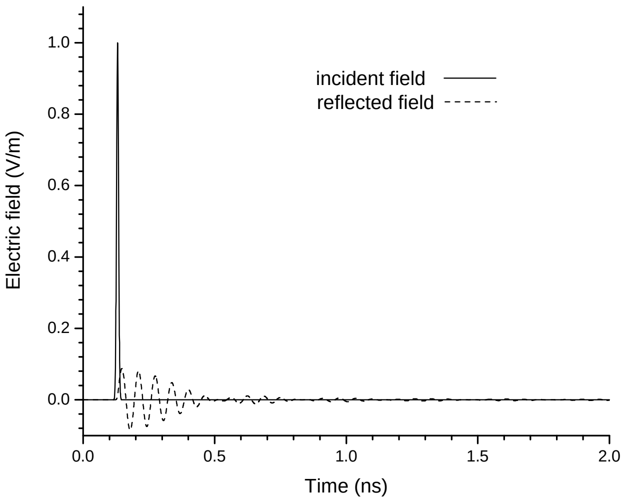

Fig. 4. The incident and reflected electric field, at a position one node in front of the air-metamaterial interface, versus time

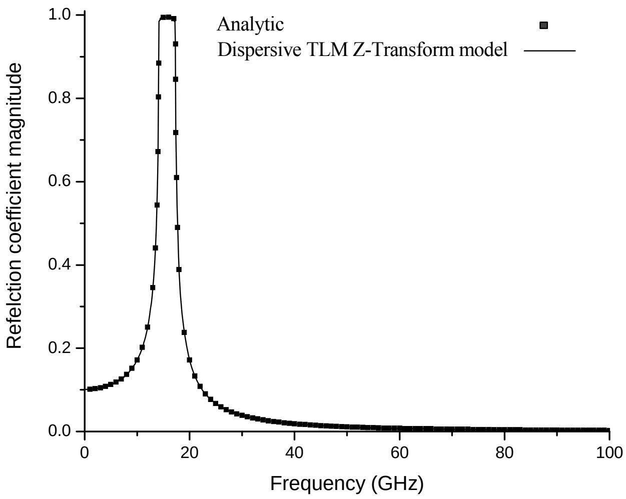

Fig. 5. Reflection coefficient magnitude of air-metamaterial interface versus frequency

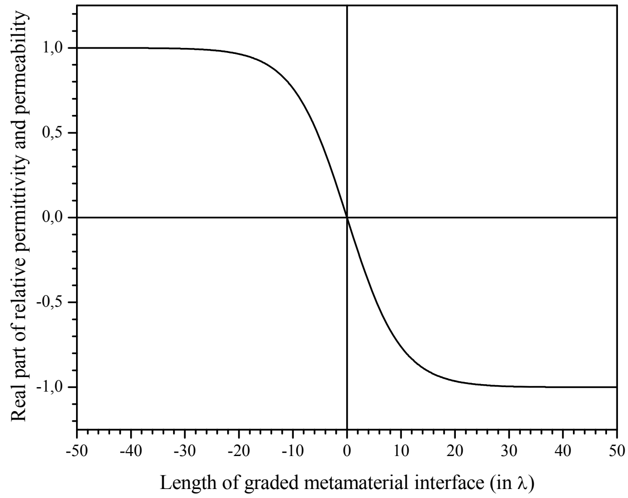

Fig. 6. Hyperbolic tangent profile for the real parts of the relative permittivity and permeability

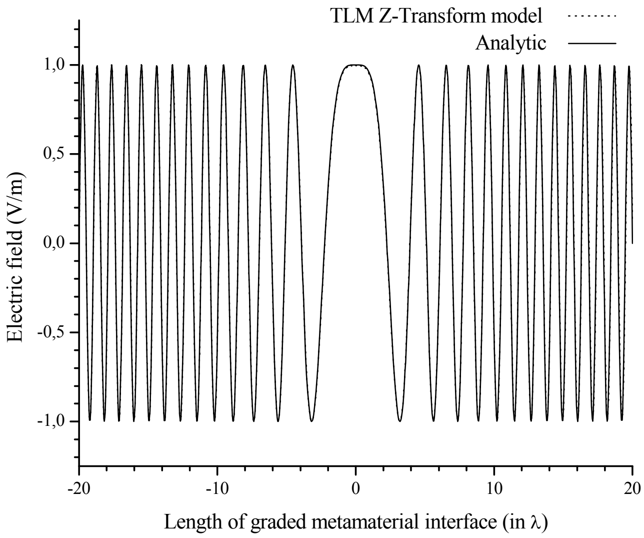

Fig. 7. Comparison of the analytical and numerical results for the lossless graded metamaterial interface

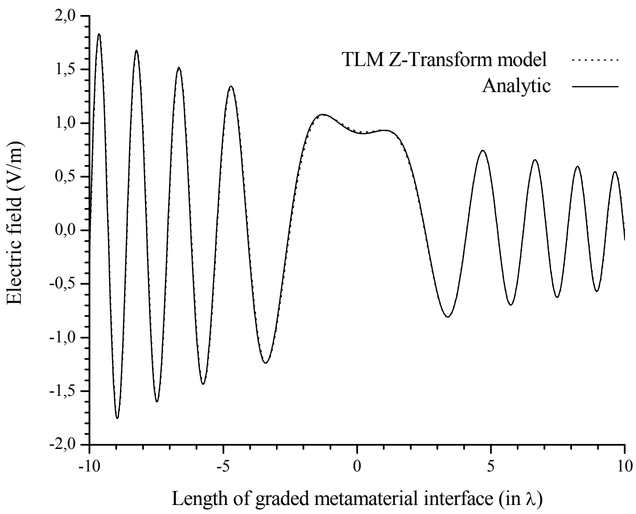

Fig. 8. Comparison of the analytical and numerical results for the lossy graded metamaterial interface

IV. Conclusion

In this paper, an extension of the Z-transform based TLM method which allows the direct time-domain modelling of metamaterials has been described. The metamaterial considered here was described using the Drude model, but the method is easily adapted to describe Lorentz or higher-order material responses. The ability of the model to account for the frequency-dispersive LH MTM behaviour was demonstratred along with its accuracy and stability. The time-domain results confirmed some of the theoretically predicted behaviour of metamaterials and gave further physical insight into the interaction of electromagnetic waves with metamaterials.

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