Lossy Wave Propagation through a Graded Interface to a Negative Index Material – Case of Constant Impedance

M. Dalarsson[1], M. Norgren[1], and Z. Jaksic[2]

Abstract – **We investigate the transmission and reflection properties at the interfaces between the lossy positive-index materials (PIM) and negative-index materials (NIM) with graded permittivity and permeability. We present an exact analytical solution to Helmholtz' equation for the oblique incidence on profile with the graded real parts of permittivity and permeability profile changing according to a hyperbolic tangent function. Thereafter we restrict our considerations to a special case of the normal incidence and present an exact analytical solution to Helmholtz' equation for a lossy case The simple analytical solutions and graphical results for the field intensity along the graded structure are presented. The model allows for arbitrary temporal dispersion.**

Keywords – **Graded Index, Metamaterials, Negative Refractive Index, Transformation Optics**

I. Introduction

During the last decade, a new paradigm in electrodynamics and electromagnetic optics of artificial composite materials has emerged. It is based on a new class of artificial composite materials with properties differing from those normally met in nature. One of the best known classes of these materials are negative index materials (NIM) with negative effective electric permittivity and negative effective magnetic permeability. NIM are often produced using "particles" – structures with subwavelength dimensions which perform a function of "atoms" of such designer materials, examples being e.g. split-ring resonators and ultrathin wires [1]. In the theoretical work of Veselago [2] it was shown that NIM materials have a negative index of refraction (and, hence, negative phase velocity), inverse Doppler effect, radiation tension instead of pressure, etc. These properties are related to the fact that the Poynting vector in these materials is antiparallel to the wavevector, i.e., the electric field, the magnetic field and the wavevector of a plane electromagnetic wave form a left-handed system of reference.

The practical implementation of NIM was proposed in the works of Pendry [3, 4], who first suggested split-ring resonators and wire arrays as the metamaterial particles. The experimental confirmation of a negative-index material was published in 2001 [5]. Split-ring resonators and nanowires are still widely used in the microwave domain and are now well understood, but many other particles such as cut-wire pairs, fishnets [1] and coupled split-ring resonators [6] have been subsequently investigated.

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Several remarkable applications of NIM have been proposed, including superlenses and hyperlenses that enable imaging below the diffraction limit [7,8], waveguides that can stop light [9], miniaturized photonic devices such as Fabry-Perot resonators and waveguides [10], and even invisibility cloaks through the technique of transformation optics [11, 12].

Structures with refractive index gradient varying from positive to negative value or cive versa were studied in the framework of metamaterial gradient index lenses by a few authors [13,14], who have shown that this provides an additional degree of design freedom. A gradient metamaterial lens was demonstrated experimentally by Smith [15]. Theoretical investigations of structures including negative index materials with graded permittivity and permeability have been done only very recently [16–18]. Other proposed applications of graded index include transformation optics and invisibility cloaks.

A simple way to practically implement gradient of negative refractive index is to utilize subwavelength structuring where the dimensions of the unit cell are continually varied. In fishnet metamaterials which are often used for high frequency NIMs, this may be done by changing the diameter of holes drilled through metal-dielectric-metal sandwich (Fig. 1) or by incrementally varying the distance between holes along one direction.

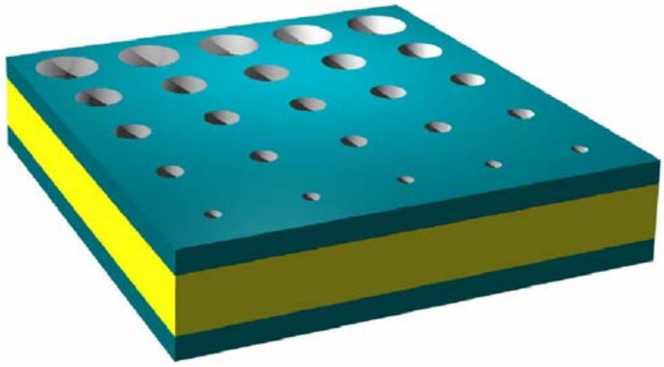


Fig. 1. Planar implementation of fishnet NIM material with refractive index gradient along a single direction. The substrate is a metal-dielectric-metal sandwich and the spatial dispersion is obtained by varying the hole diameter

In the present paper, we consider the oblique incidence of the electromagnetic waves and the transmission and reflection properties of optical structures with a graded transition from a positive-index material (PIM) to a negative-index material (NIM) or vice versa.

Furthermore we present a special case of an exact analytical solution of Helmholtz' equation for the normal-incidence propagation of electromagnetic waves through a lossy graded metamaterial structure. We choose a graded profile for which the real parts of both the permittivity and the permeability vary according to a hyperbolic tangent function.

II. Field Equations

We perform our electromagnetic analysis *ab initio*, searching for fields that vary periodically in time according to an *exp(iωt)* law. We assume that the quasistatic approximation is valid and that the material is isotropic. Thus the optical properties of structural constituents can be described by the effective dielectric permittivity and the effective magnetic permeability. For practically all metamaterials the effective medium assumption is valid, because their constituent elements are well on the subwavelength level. Let us assume the perpendicular polarization of the incoming wave, such that the electric and magnetic field vectors are given by

$$\begin{aligned} \vec{E}(\vec{r}) &= -E(x,y)\,\vec{z}_0 \quad , \\ \vec{H}(\vec{r}) &= H(x,y)\cos\theta\,\vec{y}_0 - H(x,y)\sin\theta\,\vec{x}_0 \end{aligned} \quad , \qquad (1)$$

where θ is the angle of incidence. The wave equation for the electric field is then obtained in the form

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} - \frac{1}{\mu}\frac{d\mu}{dx}\frac{\partial E}{\partial x} + \omega^2 \mu\varepsilon E(x,y) = 0 \qquad (2)$$

where $\varepsilon = \varepsilon(\omega, x)$ and $\mu = \mu(\omega\, x)$ are the frequency- and space-dependent (x-dependent) dielectric permittivity and magnetic permeability, respectively. This equation describes the propagation of an electromagnetic wave through a medium with the constitutive parameters that vary along the propagation direction. The spatial dependency of the functions $\varepsilon = \varepsilon(\omega, x)$ and $\mu = \mu(\omega\, x)$ may be completely arbitrary, even on space scales shorter than the wavelength of the radiation, on the obvious condition that the effective medium approximation remains valid.

Thus the spectral properties and the gradual transition between the two materials are described by $\varepsilon = \varepsilon(\omega, x)$ and $\mu = \mu(\omega\, x)$. Following the approach in [17]-[20], we consider an inhomogeneous medium for which the real parts of the effective permittivity and permeability vary according to a hyperbolic tangent function. The hyperbolic tangent function is chosen because it allows for simple exact analytical solutions of Eq. (2) and it allows a detailed study of the limit of the abrupt transition as well. We use the functions incorporating arbitrary losses

$$\mu = -\mu_0 \mu_R(\omega)\tanh(\rho x) - i\mu_0 \mu_I(\omega) \ , \qquad (3)$$

$$\varepsilon = -\varepsilon_0 \varepsilon_R(\omega)\tanh(\rho x) - i\varepsilon_0 \varepsilon_I(\omega) \ , \qquad (4)$$

where ρ is an arbitrary parameter describing the steepness of the smooth transition from the PIM material in $x < 0$ to the NIM material in $x > 0$. Note that for the material to be passive, it follows that $\varepsilon_I(\omega) > 0$ and that $\mu_I(\omega) > 0$.

In order to obtain a constant wave impedance throughout the structure, we require that the real and imaginary parts of the effective permittivity and permeability satisfy the following condition

$$\frac{\mu_I(\omega)}{\mu_R(\omega)} = \frac{\varepsilon_I(\omega)}{\varepsilon_R(\omega)} = \beta(\omega) \ . \qquad (5)$$

Thus we have

$$\mu = -\mu_0 \mu_R(\omega)\left[\tanh(\rho x) + \mathrm{i}\beta\right] , \qquad (6)$$

$$\varepsilon = -\varepsilon_0 \varepsilon_R(\omega)\ \left[\tanh(\rho x) + \mathrm{i}\beta\right] . \qquad (7)$$

Except for the condition (5) our method allows for arbitrary temporal dispersion. Note that the optimization condition results in wave impedance

$$Z = Z_0 Z(\omega) = \sqrt{\mu_0 \mu_R(\omega) / \varepsilon_0 \varepsilon_R(\omega)} \qquad (8)$$

being constant throughout the entire structure. Thus there is no reflection on the graded interface between the two materials.

In order to solve Eq. (2), we assume that the electric field strength can be written in the form $E(x, y) = X(x)Y(y)$. Thus we obtain two ordinary differential equations for the functions $X(x)$ and $Y(y)$ in the form

$$\begin{aligned} &\frac{d^2 X}{dx^2} - \frac{1}{\mu}\frac{d\mu}{dx}\frac{dX}{dx} + \left(\omega^2 \mu\varepsilon - k_y^2\right)X(x) = 0 \\ &\frac{d^2 Y}{dy^2} + k_y^2 Y(y) = 0 \end{aligned} \quad , \qquad (9)$$

where $k_y = k \sin\theta$. The second equation in (4) is elementary and its solution is $Y(y) = \exp(\pm k_y\, y)$. The standard approach to the solution of the first equation in (4) is to eliminate the first order terms by introducing the function $F(x)$ instead of the function $X(x)$ using the following transformation

$$X(x) = \sqrt{\mu(x)}F(x) \ . \qquad (10)$$

In this way, we obtain the following wave equation for the function $F(x)$

$$\frac{d^2 F}{dx^2} + \left[\omega^2 \mu\varepsilon + \frac{1}{2\mu}\frac{d^2\mu}{dx^2} - \frac{3}{4\mu^2}\left(\frac{d\mu}{dx}\right)^2\right]F(x) = 0 \ . \qquad (11)$$

This equation can also be written as a wave equation

$$\frac{d^2F}{dx^2}+k_\mu^2(x)F(x)=0 \quad , \tag{12}$$

where

$$k_\mu^2(x)=\omega^2\mu\varepsilon+\frac{1}{2\mu}\frac{d^2\mu}{dx^2}-\frac{3}{4\mu^2}\left(\frac{d\mu}{dx}\right)^2 \tag{13}$$

is the space-dependent effective wave vector for the electric field. In case of the hyperbolic tangent profile for the functions ε = ε(ω, x) and μ = μ(ω x) given by Eqs. (6)-(7), Eq. (13) is generally reduced to a hypergeometric equation, allowing for analytical solution in terms of suitable hypergeometric functions.

III. Solutions of the Field Equations

The exact analytical solution of Eq. (2), with the proper normalization, is given by

$$E^{\pm}(x)=E_0^{\pm}e^{\mp k\beta x}\frac{\Gamma\left(1\mp i\frac{k}{2\rho}(\cos\theta+1)\right)\Gamma\left(1\mp i\frac{k}{2\rho}(\cos\theta-1)\right)}{\Gamma(2)\Gamma(\mp i(k/\rho)\cos\theta)}$$

$$\times[2\cosh(\rho x)]^{\pm ik/\rho\cos\theta}\tanh^2\rho x \quad ,(14)$$

$$\times\, {}_2F_1\left(1\pm i\frac{k}{2\rho}(\cos\theta+1), 1\pm i\frac{k}{2\rho}(\cos\theta-1), 2;\ \tanh^2\rho x\right)$$

$$\times e^{iky\sin\theta}$$

where ${}_2F_1(a\ ,\ b\ ,\ c\ ;\ z)$ is the hypergeometric function, $E_0^{\pm}$ is the amplitude of the electric field at the boundary x = 0, and $k^2=-\omega^2\varepsilon_0\mu_0\varepsilon_R(\omega)\mu_R(\omega)$. We note that in the absence of losses (β = 0), the result (14) is reduced to the results in references [17]-[18] as special cases. For lossless media (β→0) we obtain the asymptotic expression for the electric field $E^{\pm}(x)$ in the limit $x\to-\infty$, as

$$\begin{aligned}E^{\pm}(\vec r)&\sim E_0^{\pm}e^{\mp k\beta x}\exp\left(\mp i\vec k\cdot\vec r\right)=\\&=E_0e^{\mp k\beta x}\exp\left[\mp ik(x\cos\theta+y\sin\theta)\right]\end{aligned} \tag{15}$$

which results in the time-domain field of the form ($E^{\pm}(\vec r,t)=\mathrm{Re}\{E^{\pm}(\vec r)e^{i\omega t}\}$ etc.)

$$E^{\pm}(\vec r)\sim E_0^{\pm}e^{\mp k\beta x}\cos(\omega t\mp\vec k\cdot\vec r)\ , \tag{16}$$

Analogously for lossless media (β→0) we obtain the asymptotic expression for the electric field $E^{\pm}(x)$ in the limit $x\to+\infty$, as

$$\begin{aligned}E^{\pm}(x)&\sim E_0^{\pm}e^{\mp k\beta x}\exp\left(\pm i\vec k\cdot\vec r\right)=\\&=E_0e^{\mp k\beta x}\exp\left[\pm ik(x\cos\theta+y\sin\theta)\right]\end{aligned}\ , \tag{17}$$

which results in the time-domain field of the form ($E^{\pm}(\vec r,t)=\mathrm{Re}\{E^{\pm}(\vec r)e^{i\omega t}\}$ etc.)

$$E^{\pm}(\vec r)\sim E_0^{\pm}e^{\mp k\beta x}\cos(\omega t\pm\vec k\cdot\vec r)\ , \tag{18}$$

From the asymptotic expressions (15)-(18) we see that to the left of the interface at x = 0, i.e. in the PIM material (ε > 0, μ > 0), we have an electromagnetic wave with the wave vector $\vec k_1=+\vec k$ propagating to the right. On the other side, to the right of the interface at x = 0, i.e. in the NIM material (ε < 0, μ < 0), we have an electromagnetic wave with the wave vector $\vec k_2=-\vec k$ propagating to the right.

However, the energy flux (the Poynting vector) still propagates forward in both media. The exact solution (14) is valid for arbitrary steepness ρ of the graded index interface.

In the special case of the normal incidence (θ = 0), the geometry of the problem is reduced to the one shown in Fig. 2.

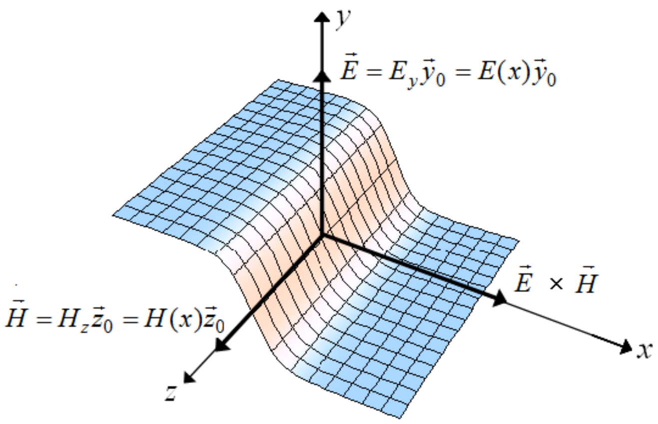


Fig. 2. Propagation of a wave through a graded indexstructure with a hyperbolic tangent profile

The equations for the fields *E(x)* and *H(x)* now become

$$\frac{d^2E}{dx^2}-\frac{1}{\mu}\frac{d\mu}{dx}\frac{dE}{dx}+\omega^2\mu\varepsilon E=0 \tag{19}$$

$$\frac{d^2H}{dx^2}-\frac{1}{\varepsilon}\frac{d\varepsilon}{dx}\frac{dH}{dx}+\omega^2\mu\varepsilon H=0 \tag{20}$$

The exact analytical solutions for the electric and magnetic fields acquire the remarkably simple forms

$$E^{\pm}(x) = E_0^{\pm} e^{\mp k\beta x} \left[2\cosh(\rho x)\right]^{\pm ik/\rho} , \tag{21}$$

$$H^{\pm}(x) = \pm H_0^{\pm} e^{\mp k\beta x} \left[2\cosh(\rho x)\right]^{\pm ik/\rho} , \tag{22}$$

where $E_0^{\pm}$ and $H_0^{\pm}$ are the amplitudes of the electric and magnetic fields at x = 0, and $k^2 = -\omega^2 \varepsilon_0 \mu_0 \varepsilon_R(\omega) \mu_R(\omega)$. In the absence of losses (β = 0), the results (21)-(22) are reduced to the results in reference [17] as a special case. The field amplitudes are related by $E_0^{\pm} = Z_0 Z(\omega) H_0^{\pm}$. The exact solutions (21)-(22) are valid for arbitrary steepness ρ of the graded index interface and arbitrary losses β. In the positive index material (PIM), we obtain for $x \to -\infty$ that

$$E^{\pm}(x) \sim E_0^{\pm} e^{\mp k\beta x} e^{\mp ikx} , \tag{23}$$

$$H^{\pm}(x) \sim \pm H_0^{\pm} e^{\mp k\beta x} e^{\mp ikx} , \tag{24}$$

which results in the time-domain fields of the form
($E(x, t) = \mathrm{Re}\{E(x)e^{i\omega t}\}$etc.)

$$E^{\pm}(x) \sim E_0^{\pm} e^{\mp k\beta x} \cos(\omega t \mp kx) , \tag{25}$$

$$H^{\pm}(x) =\sim H_0^{\pm} e^{\mp k\beta x} \cos(\omega t \mp kx) , \tag{26}$$

In the LHM material, we obtain for $x \to +\infty$ that

$$E^{\pm}(x) \sim E_0^{\pm} e^{\mp k\beta x} e^{\pm ikx} , \tag{27}$$

$$H^{\pm}(x) \sim \pm H_0^{\pm} e^{\mp k\beta x} e^{\pm ikx} , \tag{28}$$

resulting in

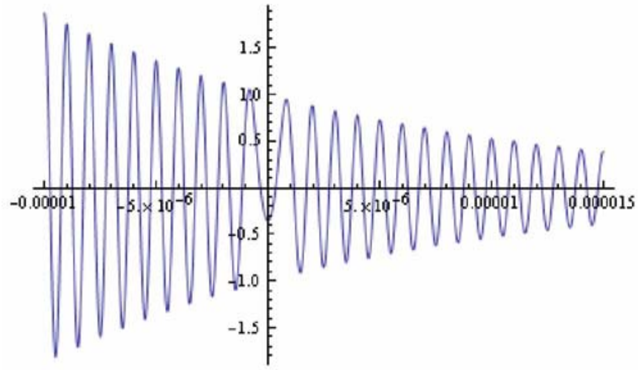
$$E^{\pm}(x) \sim E_0^{\pm} e^{\mp k\beta x} \cos(\omega t \mp (-k)x) , \tag{29}$$

$$H^{\pm}(x) =\sim H_0^{\pm} e^{\mp k\beta x} \cos(\omega t \mp (-k)x) , \tag{30}$$

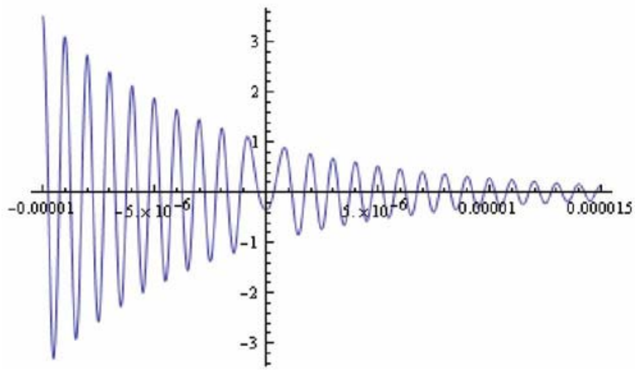
For $x \to -\infty$, it follows from the results (25)-(26) that the solution with the upper superscript has the wavevector $\vec{k}_{RHM} = +k\,\vec{x}_0$, i.e. the wave propagates in the +x-direction. On the other hand for $x \to +\infty$ it follows from the results (29)-(30) that in the LHM this wave has the wavevector $\vec{k}_{LHM} = -k\,\vec{x}_0$, i.e. the wave propagates in the −x-direction. However, the energy flux (the Poynting vector) is still in the +x-direction in both media. Conversely, the solution with the lower superscript is a wave that propagates in the +x-direction in the LHM, in the −x-direction in the RHM, and transports energy in the −x-direction in both media.

IV. Graphical Presentation and Discussion of the Results

The exact analytical solutions for the real part of the electric field E(x), given by Eq. (21), for two different values of the numerical parameters are presented in Fig. 3. From Fig. 3, we see that in this particular case, there is no reflection at the interface between a PIM and a NIM material. This is expected, since in our case the impedance is constant throughout the entire space. From the curves presented in Fig. 3, we see that for $\beta = 10^{-2}$ there is a moderate attenuation of the signal over the considered distance. For $\beta = 2\times10^{-2}$, we see that there is a strong attenuation and the signal gets much weaker.



(a)



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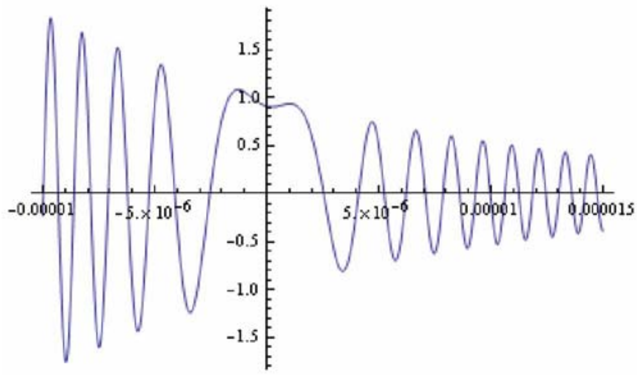
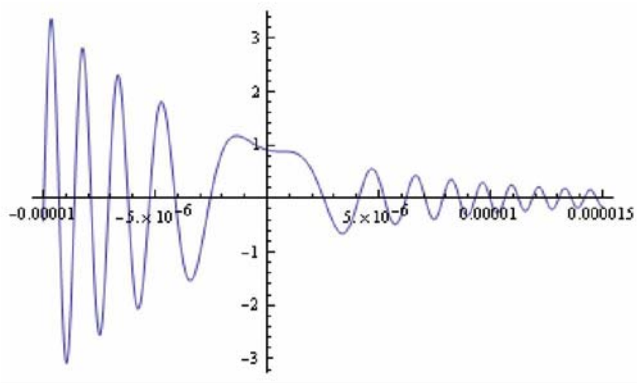


(c)



(d)

Fig. 3. Analytical results for the real part of the electric field $E(x)$ as a function of x, with $E_0 = 1$ and $k = 2\pi/(10^{-6}\text{m})$.
a) $\rho = 1/(10^{-6}\text{m})$ and $\beta = 10^{-2}$.
b) $\rho = 1/(10^{-6}\text{m})$ and $\beta = 2 \times 10^{-2}$.
c) $\rho = 1/(10^{-5}\text{m})$ and $\beta = 10^{-2}$.
d) $\rho = 1/(10^{-5}\text{m})$ and $\beta = 2 \times 10^{-2}$.

This indicates that in any practical application the loss factor β must be much smaller than unity ($\beta << 1$) and certainly considerably smaller than 10^{-2}. However, our exact analytical model does not require any such restrictions and an arbitrary loss factor β is allowed.

V. Conclusion

We have presented an exact analytical solution to Helmholtz' equation for a lossy case with graded permittivity and permeability profile changing according to a hyperbolic tangent function. The expressions for the field intensity in the graded structure have been derived for both the oblique incidence with perpendicular polarization and for the special case of the normal incidence. The graphical results for the field intensity in the case of normal incidence have been presented. The model allows for arbitrary temporal dispersion and arbitrary losses.

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